

# A class of asymmetric gapped Hamiltonians on quantum spin chains and its characterization III

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## Abstract

In this paper, we consider classification problem of asymmetric gapped Hamiltonians, which are given as the non-degenerate part of the Hamiltonians introduced in [O1]. We consider the  $C^1$ -classification, which takes into account the effect of boundaries. We show that the left and right degeneracies of edge ground states are the complete invariant. As a corollary, we consider the bulk-classification problem. We study Hamiltonians that 1. are given by translation invariant finite range interactions, 2. are gapped in the bulk, 3. are frustration-free, 4. have uniformly bounded ground state degeneracy on finite intervals, and 5. have a unique bulk ground state. We show that for the bulk-classification, any such Hamiltonians are equivalent.

## 1 Introduction

In Part I [O1], we introduced a class of gapped Hamiltonians on quantum spin chains, which allows asymmetric edge ground states. They are defined as MPS (matrix product state) Hamiltonians given by  $n$ -tuple of matrices in ClassA, a class we introduced in [O1]. It is an asymmetric generalization of the class of Hamiltonians given in [FNW]. We investigated the properties of this new class in Part I [O1]. In particular, we showed that Hamiltonians in this class satisfies five qualitative physical conditions, which are listed as [A1]-[A5] in [O2]. We call  $\tilde{H}$ , the set of Hamiltonians satisfying [A1]-[A5]. In Part II [O2], conversely, we showed that these five properties [A1]-[A5] actually guarantee the ground state structure of the Hamiltonian to be captured by the MPS Hamiltonians given by ClassA. More precisely, we showed for any Hamiltonian in  $\tilde{H}$ , there is an MPS Hamiltonian given by ClassA satisfying the followings: 1. The ground state spaces of the two Hamiltonians on the infinite intervals coincide. 2. The spectral projections onto the ground state space of the original Hamiltonian on finite intervals are well approximated by that of the MPS one. From the latter property we see that two Hamiltonians are in the same class in the classification problem of gapped Hamiltonians with open boundary conditions, (the type II- $C^1$ -classification). Hence the classification problem of  $\tilde{H}$  is reduced to the classification problem of MPS Hamiltonians given by ClassA. In this paper, we classify MPS Hamiltonians given by ClassA', where ClassA' is the “non-degenerate” part of ClassA. We show that the left and right degeneracies of edge ground states are the complete invariant for the type II- $C^1$ -classification, for this class.

As an important application, we consider bulk-classification problem of gapped Hamiltonians. It is believed that there exists only one gapped bulk ground state phase in one dimensional quantum spin systems. It has been an open problem for a while, if two MPS-Hamiltonians from the class of [FNW] can be connected to each other without closing the gap *in the bulk*, and *without breaking*

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the translation invariance ([BO],[SW]). We solve this affirmatively. More generally, we consider Hamiltonians which 1. are given by translation invariant finite range interactions, 2. are gapped in the bulk, 3. are frustration-free, 4. have uniformly bounded ground state degeneracy on finite intervals, and 5. have a unique bulk ground state,. We show that for the bulk-classification, any such Hamiltonians are equivalent.

Throughout this article,  $2 \leq n \in \mathbb{N}$  is fixed as the dimension of the spin under consideration. We use freely the notations and definitions given in Part I [O1] Subsection 1.1, 1.2, 1.3, and Appendix A.

### 1.1 Class $A'$ and its type II- $C^1$ -classification

For  $m \in \mathbb{N}$ , we denote by  $\mathcal{J}_m$  the set of all positive translation invariant interactions with interaction length less than or equal to  $m$ . We also set  $\mathcal{J} := \cup_{m \in \mathbb{N}} \mathcal{J}_m$ . A natural number  $m \in \mathbb{N}$  and an element  $h \in \mathcal{A}_{[0, m-1]}$ , define an interaction  $\Phi_h$  by

$$\Phi_h(X) := \begin{cases} \tau_x(h), & \text{if } X = [x, x + m - 1] \text{ for some } x \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

for  $X \in \mathfrak{S}_{\mathbb{Z}}$ . A Hamiltonian associated with  $\Phi$  is a net of self-adjoint operators  $H_\Phi := ((H_\Phi)_\Lambda)_{\Lambda \in \mathcal{J}_{\mathbb{Z}}}$  such that

$$(H_\Phi)_\Lambda := \sum_{X \subset \Lambda} \Phi(X). \quad (2)$$

First we confirm what we mean by gapped Hamiltonian with open boundary conditions.

**Definition 1.1.** Let  $\Phi$  be a translation invariant finite range interaction on  $\mathcal{A}_{\mathbb{Z}}$ . For each  $N \in \mathbb{N}$ , let  $G_{[0, N-1]}$  be the spectral projection of  $(H_\Phi)_{[0, N-1]}$  corresponding to the lowest eigenvalue  $\inf \sigma((H_\Phi)_{[0, N-1]})$ . We say that the Hamiltonian  $H_\Phi$  is gapped with respect to the open boundary condition if there exist  $\gamma > 0$  and  $N_0 \in \mathbb{N}$  such that

$$\inf \sigma((H_\Phi)_{[0, N-1]}) + \gamma (\mathbb{I} - G_{[0, N-1]}) \leq (H_\Phi)_{[0, N-1]}, \quad N_0 \leq N \in \mathbb{N}.$$

We call this  $\gamma$ , a lower bound of the gap.

Let us recall the definition of  $C^1$ -classification I, II.

**Definition 1.2** ( $C^1$ -classification I of gapped Hamiltonians). Let  $H_0, H_1$  be Hamiltonians gapped with respect to the open boundary conditions, associated with interactions  $\Phi_0, \Phi_1 \in \mathcal{J}$ . We say that  $H_0, H_1$  are type I- $C^1$ -equivalent if the following conditions are satisfied.

1. There exist an  $m \in \mathbb{N}$  and a continuous and piecewise  $C^1$ -path  $\Phi : [0, 1] \rightarrow \mathcal{J}_m$  such that  $\Phi(0) = \Phi_0, \Phi(1) = \Phi_1$ .
2. Let  $H(t)$  be the Hamiltonian associated with  $\Phi(t)$  for each  $t \in [0, 1]$ . There are  $\gamma > 0$ ,  $N_0 \in \mathbb{N}$ , and finite intervals  $I(t) = [a(t), b(t)]$ , whose endpoints  $a(t), b(t)$  smoothly depending on  $t \in [0, 1]$ , such that for any  $N_0 \leq N \in \mathbb{N}$ , the smallest eigenvalue of  $H(t)_{[0, N-1]}$  is in  $I(t)$  and the rest of the spectrum is in  $[b(t) + \gamma, \infty)$ .

We write  $H_0 \simeq_I H_1$  when  $H_0, H_1$  are type I- $C^1$ -equivalent.

**Definition 1.3** ( $C^1$ -classification II of gapped Hamiltonians). Let  $H_0, H_1$  be Hamiltonians gapped with respect to the open boundary conditions, associated with interactions  $\Phi_0, \Phi_1 \in \mathcal{J}$ . We say that  $H_0, H_1$  are type II- $C^1$ -equivalent if the following conditions are satisfied.

1. There exist  $m \in \mathbb{N}$  and a continuous and piecewise  $C^1$ -path  $\Phi : [0, 1] \rightarrow \mathcal{J}_m$  such that  $\Phi(0) = \Phi_0, \Phi(1) = \Phi_1$ .

2. Let  $H(t)$  be the Hamiltonian associated with  $\Phi(t)$  for each  $t \in [0, 1]$ . There are  $\gamma > 0$ ,  $N_0 \in \mathbb{N}$ , and finite intervals  $I(t) = [a(t), b(t)]$ ,  $t \in [0, 1]$ , satisfying the followings:
- (i) the endpoints  $a(t), b(t)$  smoothly depend on  $t \in [0, 1]$ , and
  - (ii) there exists a sequence  $\{\varepsilon_N\}_{N \in \mathbb{N}}$  of positive numbers with  $\varepsilon_N \rightarrow 0$ , for  $N \rightarrow \infty$ , such that  $\sigma(H(t)_{[0, N-1]}) \cap I(t) = \sigma(H(t)_{[0, N-1]}) \cap [\lambda(t, N), \lambda(t, N) + \varepsilon_N]$ , and  $\sigma(H(t)_{[0, N-1]}) \cap I(t)^c = \sigma(H(t)_{[0, N-1]}) \cap [b(t) + \gamma, \infty)$  for all  $N \geq N_0$  and  $t \in [0, 1]$ , where  $\lambda(t, N)$  is the smallest eigenvalue of  $H(t)_{[0, N-1]}$ .

We write  $H_0 \simeq_{II} H_1$ , when  $H_0$  and  $H_1$  are *typeII* –  $C^1$ -equivalent.

In Part II [O2], we discussed the type II- $C^1$ -classification for a class of Hamiltonians  $\tilde{\mathcal{H}}$  satisfying five qualitative properties. We showed that any element in  $\tilde{\mathcal{H}}$  is type II- $C^1$ -equivalent to an MPS Hamiltonian  $H_{\Phi_m, \mathbb{B}}$  which is given by some  $\mathbb{B} \in \text{ClassA}$ , the class which was introduced in Part I [O1]. In other words, the problem of type II- $C^1$ -classification for this family  $\tilde{\mathcal{H}}$  is reduced to the classification problem of MPS Hamiltonians given by ClassA. In this paper, we classify Hamiltonians given by the “non-degenerate” part of ClassA, which we call ClassA’.

The subclass ClassA’ is the set of elements of ClassA with the non-degeneracy condition on  $\lambda$ . Namely we assume  $\lambda$  in Definition 1.14 Part I [O1] to be in  $\text{Wo}'(K_R, k_L)$  defined in the following.

**Definition 1.4.** For  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , we denote by  $\text{Wo}'(k_R, k_L)$  the set of all  $\lambda = (\lambda_{-k_R}, \dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots, \lambda_{k_L}) \in \mathbb{C}^{k_R+k_L+1}$  satisfying

$$\lambda_0 = 1, \quad 0 < |\lambda_{-k_R}| \leq |\lambda_{-k_R+1}| \leq \dots |\lambda_{-1}| < 1, \quad 0 < |\lambda_{k_L}| \leq |\lambda_{k_L-1}| \leq \dots |\lambda_1| < 1,$$

and  $\lambda_i \neq \lambda_j$  if  $-k_R \leq i < j \leq 0$  or  $0 \leq i < j \leq k_L$ .

The last condition in the definition corresponds to the non-degeneracy. Recall the definitions in subsection 1.3 of [O1]. In particular, ClassA is defined as follows.

**Definition 1.5.** Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$  and  $(\lambda, \mathbb{D}, \mathbb{G}, Y) \in \mathcal{T}(k_R, k_L)$ . We denote by  $\mathfrak{B}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$  the set of all  $n$ -tuples  $\mathbb{B} = (B_1, \dots, B_n) \in M_{n_0} \otimes (\mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) \Lambda_\lambda (1 + Y))$  satisfying

$$l_{\mathbb{B}} = l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y) := \inf \left\{ l \mid \mathcal{K}_{l'}(\mathbb{B}) = M_{n_0} \otimes \left( \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) (\Lambda_\lambda (1 + Y))^{l'} \right) \text{ for all } l' \geq l \right\} < \infty, \quad (3)$$

and  $r_{T_{\mathbb{B}}} = 1$ . We define ClassA by

$$\text{ClassA} := \bigcup \{ \mathfrak{B}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y) \mid n_0 \in \mathbb{N}, k_R, k_L \in \mathbb{N} \cup \{0\}, (\lambda, \mathbb{D}, \mathbb{G}, Y) \in \mathcal{T}(k_R, k_L) \}.$$

We define ClassA’, as the “non-degenerate” part of ClassA. The non-degeneracy is about  $\lambda$ , i.e.,  $\lambda$  belongs to  $\text{Wo}'(k_R, k_L)$ , the non-degenerate part  $\text{Wo}(k_R, k_L)$ .

**Definition 1.6.** We define ClassA’ by

$$\begin{aligned} \text{ClassA}' &:= \bigcup \{ \mathfrak{B}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y) \mid n_0 \in \mathbb{N}, k_R, k_L \in \mathbb{N} \cup \{0\}, (\lambda, \mathbb{D}, \mathbb{G}, Y) \in \mathcal{T}(k_R, k_L), \lambda \in \text{Wo}'(k_R, k_L) \} \\ &= \bigcup \{ \mathfrak{B}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, 0) \mid n_0 \in \mathbb{N}, k_R, k_L \in \mathbb{N} \cup \{0\}, (\lambda, \mathbb{D}, \mathbb{G}, 0) \in \mathcal{T}(k_R, k_L), \lambda \in \text{Wo}'(k_R, k_L) \}. \end{aligned}$$

*Remark 1.7.* The condition  $\lambda \in \text{Wo}'(k_R, k_L)$  and  $(\lambda, \mathbb{D}, \mathbb{G}, Y) \in \mathcal{T}(k_R, k_L)$  automatically implies  $Y = 0$  (Lemma 3.2). This corresponds to the second inequality.

Furthermore, for each  $n_0 \in \mathbb{N}$  and  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , we set

$$\text{Class}(n, n_0, k_R, k_L) := \bigcup \{ \mathfrak{B}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, 0) \mid (\lambda, \mathbb{D}, \mathbb{G}, 0) \in \mathcal{T}(k_R, k_L), \lambda \in \text{Wo}'(k_R, k_L) \}.$$

We denote by  $\mathcal{H}(n, n_0, k_R, k_L)$ , the set of Hamiltonians  $H_{\Phi_{m, \mathbb{B}}}$  given by  $\mathbb{B} \in \text{Class}(n, n_0, k_R, k_L)$ , and  $m \geq 2l_{\mathbb{B}}$ . Recall from Part I [O1] Theorem 1.18 (vi) that  $n_0(k_R + 1)$  (resp.  $n_0(k_L + 1)$ ) is the degree of ground state degeneracy on right (resp. left) half infinite chain. The following theorem says that these numbers are complete invariant of the type II  $C^1$ -classification of MPS Hamiltonians given by  $\text{ClassA}'$ .

**Theorem 1.8.** *Let  $n_0, n'_0 \in \mathbb{N}$ ,  $k_R, k_L, k'_R, k'_L \in \mathbb{N} \cup \{0\}$ ,  $\mathbb{B} \in \text{Class}(n, n_0, k_R, k_L)$ , and  $\mathbb{B}' \in \text{Class}(n, n'_0, k'_R, k'_L)$ . Then for any  $m \geq 2l_{\mathbb{B}}$ , and  $m' \geq 2l_{\mathbb{B}'}$ , we have  $H_{\Phi_{m, \mathbb{B}}} \simeq_{II} H_{\Phi_{m', \mathbb{B}'}}$  if and only if*

$$n_0(k_R + 1) = n'_0(k'_R + 1), \text{ and } n_0(k_L + 1) = n'_0(k'_L + 1).$$

## 1.2 Bulk Classification

Let us recall the definition of a ground state of  $C^*$ -dynamical system [BR96]. Let  $\mathfrak{A}$  be a unital  $C^*$ -algebra and  $\alpha$  a strongly continuous one parameter group on  $\mathfrak{A}$ . We denote the generator of  $\alpha$  by  $\delta_\alpha$ . A state  $\omega$  on  $\mathfrak{A}$  is called an  $\alpha$ -ground state if the inequality  $-i\omega(A^*\delta_\alpha(A)) \geq 0$  holds for any element  $A$  in the domain  $\mathcal{D}(\delta_\alpha)$  of  $\delta_\alpha$ . Let  $\omega$  be an  $\alpha$ -ground state, with the GNS triple  $(\mathcal{H}, \pi, \Omega)$ . Then there exists a unique positive operator  $H_{\omega, \alpha}$  on  $\mathcal{H}$  such that  $e^{itH_{\omega, \alpha}}\pi(A)\Omega = \pi(\alpha_t(A))\Omega$ , for all  $A \in \mathfrak{A}$  and  $t \in \mathbb{R}$ . Note that  $\Omega$  is an eigenvector of  $H_{\omega, \alpha}$  with eigenvalue 0. We say that  $H_{\omega, \alpha}$  has a spectral gap if 0 is a non-degenerate eigenvalue of  $H_{\omega, \alpha}$  and there exists  $\gamma > 0$  such that  $\sigma(H_{\omega, \alpha}) \setminus \{0\} \subset [\gamma, \infty)$ . (Here,  $\sigma(H_{\omega, \alpha})$  denotes the spectrum of  $H_{\omega, \alpha}$ .)

We consider  $C^*$ -dynamical systems given by translation invariant finite range interactions on the quantum spin chain  $\mathcal{A}_{\mathbb{Z}}$ . Let  $\Phi$  be a translation invariant finite range interaction. Our  $C^*$ -dynamics is given by  $\alpha_{\Phi, t}(A) := \lim_{\Lambda \rightarrow \mathbb{Z}} e^{it(H_\Phi)_\Lambda} A e^{-it(H_\Phi)_\Lambda}$ ,  $A \in \mathcal{A}_{\mathbb{Z}}$ ,  $t \in \mathbb{R}$  with  $(H_\Phi)_\Lambda$  in (2). We denote the set of all  $\alpha_\Phi$ -ground states on  $\mathcal{A}_{\mathbb{Z}}$  by  $\mathfrak{B}_\Phi$ . Recall  $\mathcal{S}_{\mathbb{Z}}(H_\Phi)$  from subsection 1.1 of [O1]. It is known that  $\emptyset \neq \mathcal{S}_{\mathbb{Z}}(H_\Phi) \subset \mathfrak{B}_\Phi$  (See Proposition 5.3.25 of [BR96]). For each  $\varphi \in \mathfrak{B}_\Phi$ , we call the positive operator  $H_{\varphi, \alpha_\Phi}$  the bulk Hamiltonian associated to  $\varphi$ ,  $\Phi$ . We consider the following class of the interactions.

**Definition 1.9.** We denote by  $\mathcal{J}_B$ , the set of all  $\Phi \in \mathcal{J}$  which satisfy the following conditions.

1. For any  $\varphi \in \mathfrak{B}_\Phi$ , 0 is the non-degenerate eigenvalue of the bulk Hamiltonian  $H_{\varphi, \alpha_\Phi}$ .
2. There is a constant  $\gamma > 0$ , such that

$$\sigma(H_{\varphi, \alpha_\Phi}) \setminus \{0\} \subset [\gamma, \infty),$$

for any  $\varphi \in \mathfrak{B}_\Phi$ .

Now we specify what we mean by bulk-classification in this paper.

**Definition 1.10** (Bulk Classification). Let  $\Phi_0, \Phi_1 \in \mathcal{J}_B$ . We say that the Hamiltonians  $H_{\Phi_0}, H_{\Phi_1}$  are bulk equivalent if the following conditions are satisfied.

1. There exist  $m \in \mathbb{N}$  and a continuous path of interactions  $\Phi : [0, 1] \rightarrow \mathcal{J}_m \cap \mathcal{J}_B$  such that  $\Phi(0) = \Phi_0$ ,  $\Phi(1) = \Phi_1$ .
2. There is a constant  $\gamma > 0$ , such that

$$\sigma(H_{\varphi_s, \alpha_{\Phi(s)}}) \setminus \{0\} \subset [\gamma, \infty),$$

for any  $s \in [0, 1]$  and  $\varphi_s \in \mathfrak{B}_{\Phi(s)}$ .

When  $H_{\Phi_0}, H_{\Phi_1}$  are bulk equivalent, we write  $H_{\Phi_0} \simeq_B H_{\Phi_1}$ .

The class we bulk-classify is the class of frustration free Hamiltonians whose degrees of local ground states degeneracy are uniformly bounded.

**Definition 1.11.** Let  $\Phi$  be a positive translation invariant finite range interaction on  $\mathcal{A}_{\mathbb{Z}}$ . We say  $H_{\Phi}$  is frustration free if  $1 \leq \dim \ker (H_{\Phi})_{[0, N-1]}$  for all  $N \in \mathbb{N}$ . Furthermore, we say that the ground state degeneracy of  $H_{\Phi} := ((H_{\Phi})_{\Lambda})_{\Lambda \in \mathcal{I}_{\mathbb{Z}}}$  is uniformly bounded if  $\sup_N \dim \ker (H_{\Phi})_{[0, N-1]} < \infty$ . We denote by  $\mathcal{J}_{FB}$ , the set of all interactions  $\Phi \in \mathcal{J}_B$  satisfying the followings:

1. There exists a unique  $\alpha_{\Phi}$ -ground state  $\omega$  on  $\mathcal{A}_{\mathbb{Z}}$ .
2. The Hamiltonian  $H_{\Phi}$  is frustration free and the ground state degeneracy of the corresponding local Hamiltonians is uniformly bounded.

We prove the following.

**Theorem 1.12.** For any  $\Phi_0, \Phi_1 \in \mathcal{J}_{FB}$ , we have  $H_{\Phi_0} \simeq_B H_{\Phi_1}$ .

## 2 Paths of gapped Hamiltonians

In this section, we introduce a sufficient condition for a path of Hamiltonians to be uniformly gapped. We first introduce a set of conditions on sequences of subspaces.

**Definition 2.1** (*Condition 5*). Let  $n, a \in \mathbb{N}$ . Let  $\xi_j^l : [0, 1] \rightarrow \bigotimes_{i=0}^{l-1} \mathbb{C}^n$  be a map given for each  $l \in \mathbb{N}$  and  $j = 1, \dots, a$ . For each  $l \in \mathbb{N}$  and  $t \in [0, 1]$ , let  $\mathcal{D}_l(t)$  be the subspace of  $\bigotimes_{i=0}^{l-1} \mathbb{C}^n$  spanned by  $\{\xi_j^l(t)\}_{j=1}^a$ , and  $G_l(t)$  the orthogonal projection onto  $\mathcal{D}_l(t)$ . We say  $\{\xi_j^l\}_{j=1, \dots, a, l \in \mathbb{N}}$  satisfies the *Condition 5* if the following conditions are satisfied.

- (i) For any  $l \in \mathbb{N}$  and  $j = 1, \dots, a$ , the map  $\xi_j^l : [0, 1] \rightarrow \bigotimes_{i=0}^{l-1} \mathbb{C}^n$  is continuous and piecewise  $C^{\infty}$ .
- (ii) There exists an  $l' \in \mathbb{N}$  such that for all  $l \geq l'$  and  $t \in [0, 1]$ , the vectors  $\{\xi_j^l(t)\}_{j=1}^a$  are linearly independent.
- (iii) There exists an  $m' \in \mathbb{N}$  such that for all  $N \geq m' + 1$  and  $t \in [0, 1]$ ,

$$\mathcal{D}_N(t) = (\mathcal{D}_{N-1}(t) \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes \mathcal{D}_{N-1}(t)). \quad (4)$$

- (iv) There exists an  $l'' \in \mathbb{N}$  such that for all  $l \geq l''$ , there exists  $0 < \varepsilon_l < \frac{1}{2\sqrt{l}}$  such that

$$\sup_{t \in [0, 1]} \|(\mathbb{I}_{[0, N-l]} \otimes G_l(t)) (G_N(t) \otimes \mathbb{I}_{\{N\}} - G_{N+1}(t))\| < \varepsilon_l, \quad (5)$$

for all  $N \geq 2l$ .

We say  $\{\xi_j^l\}_{j=1}^a$  satisfies the *Condition 5* for  $(l', m', l'')$  when we would like to specify the numbers.

*Remark 2.2.* Suppose that  $\{\xi_j^l\}_{j=1}^a$  satisfies *Condition 5*. As the dimensions of  $\mathcal{D}_N(t)$ s are uniformly bounded by  $a$ , (iii) of *Condition 5* implies  $\mathcal{D}_m(t) \neq \bigotimes_{i=0}^{m-1} \mathbb{C}^n$ , for any  $m \geq m'$ , i.e.,  $\dim \mathcal{D}_m(t) < n^m$ , for all  $m \geq m'$ . In particular, we have  $a = \dim \mathcal{D}_{\max\{l', m'\}}(t) < n^{\max\{l', m'\}}$ . For the same reason, for  $\mathbf{v} \in \mathbb{M}_k^{\times n}$  with  $m_{\mathbf{v}} < \infty$ , we have  $\dim \mathcal{G}_{m, \mathbf{v}} < n^m$  for all  $m \geq m_{\mathbf{v}}$ . In particular, if  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ , then we have  $n_0^2(k_R + 1)(k_L + 1) < n^{2l_{\mathbb{B}}}$  by Proposition 3.1 of [O1]. Therefore, in the statement of Theorem 1.18 of [O1], we note  $2l_{\mathbb{B}} = \max\{2l_{\mathbb{B}}, \frac{\log(n_0^2(k_R+1)(k_L+1)+1)}{\log n}\}$ .

**Lemma 2.3.** Let  $n, a \in \mathbb{N}$  with  $2 \leq n$ , and  $\xi_j^l : [0, 1] \rightarrow \bigotimes_{i=0}^{l-1} \mathbb{C}^n$  be maps given for each  $l \in \mathbb{N}$  and  $j = 1, \dots, a$ . For each  $l \in \mathbb{N}$  and  $t \in [0, 1]$ , let  $\mathcal{D}_l(t)$  be the subspace of  $\bigotimes_{i=0}^{l-1} \mathbb{C}^n$  spanned by  $\{\xi_j^l(t)\}_{j=1}^a$  and  $G_l(t)$  the orthogonal projection onto  $\mathcal{D}_l(t)$ . Suppose that  $\{\xi_j^l\}_{j=1, \dots, a, l \in \mathbb{N}}$  satisfies *Condition 5* for  $(l', m', l'')$ . Then we have the followings.

1. For any  $l \geq l'$ , the map  $G_l : [0, 1] \rightarrow \bigotimes_{i=0}^{l-1} M_n(\mathbb{C})$  is continuous and piecewise  $C^\infty$ .
2. For any  $t \in [0, 1]$ ,  $\mathbf{m}_{\{\mathcal{D}_N(t)\}} \leq m'$ , (see Definition 1.3 of Part I [O1] for  $\mathbf{m}_{\{\mathcal{D}_N(t)\}}$ ).
3. For any  $m \geq m'$  and  $t \in [0, 1]$ ,  $\ker (H_{\Phi_{1-G_m(t)}})_{[0, N-1]} = \mathcal{D}_N(t)$  for all  $N \geq m$ .
4. For any  $m \geq m'$  and  $l \geq \max\{l', m\}$ ,

$$\gamma_l^m := \inf_{t \in [0, 1]} d_{\mathbb{R}} \left( \sigma \left( (H_{\Phi_{1-G_m(t)}})_{[0, l-1]} \right) \setminus \{0\}, \{0\} \right) > 0.$$

(Here,  $d_{\mathbb{R}}(A, B)$  denotes the Euclidean distance between  $A, B \subset \mathbb{R}$ .)

5. For any  $m \geq m'$ , and  $t \in [0, 1]$ ,

$$\frac{\gamma_l^m \wedge \gamma_{2l}^m}{4(l+2)} (1 - G_N(t)) \leq (H_{\Phi_{1-G_m(t)}})_{[0, N-1]}, \quad \text{for all } l \geq \max\{l'', l', m\}, \text{ and } N \geq 2l + 1.$$

6. For any  $m_0, m_1 \geq m'$ , we have  $H_{\Phi_{1-G_{m_0}(0)}} \simeq_I H_{\Phi_{1-G_{m_1}(1)}}$ .

**Proof.** 1. is immediate from Lemma A.1. It is clear that (iii) implies Property (I,  $m'$ ), namely 2 holds. (Recall (3) of [O1] for Property (I,  $m'$ .) Property (I,  $m'$ ) implies 3. Applying Lemma A.6 to  $(H_{\Phi_{1-G_m(t)}})_{[0, l-1]}$ , we obtain 4. (Here, we used  $a < n^{\max\{l', m'\}}$  to guarantee  $\text{rank} (H_{\Phi_{1-G_m(t)}})_{[0, l-1]} = n^l - a \in \mathbb{N}$ , for  $l \geq \max\{l', m'\}$ .) Theorem 3 of [N] gives 5. The properties 1.-5. implies  $H_{\Phi_{1-G_m(0)}} \simeq_I H_{\Phi_{1-G_m(1)}}$  for  $m \geq m'$ . By the argument of Lemma 4.2 [BO], we obtain

$$H_{\Phi_{1-G_{m_0}(0)}} \simeq_I H_{\Phi_{1-G_m(0)}}, \quad H_{\Phi_{1-G_{m_1}(1)}} \simeq_I H_{\Phi_{1-G_m(1)}}.$$

Hence we obtain  $H_{\Phi_{1-G_{m_0}(0)}} \simeq_I H_{\Phi_{1-G_{m_1}(1)}}$ .  $\square$

Recall the definition of  $M_{\mathbf{v}, p, q}$  given in (8), and Condition 2,3,4 given by Definition 2.3, 2.4, 2.5 of Part I [O1]

**Proposition 2.4.** Let  $n, k, m_1, m_2, m_3 \in \mathbb{N}$  and  $p, q \in \mathcal{P}(M_k)$ . Let  $v_\mu : [0, 1] \rightarrow M_k$ ,  $\mu = 1, \dots, n$  be  $C^\infty$ -maps. Let  $\{x_i\}_{i=1}^{\text{rank } p - \text{rank } q}$  be a basis of  $pM_kq$ . Suppose that for each  $t \in [0, 1]$ , the pentad  $(n, k, p, q, \mathbf{v}(t))$  satisfies the Condition 2, and the Condition 3 for  $m_1$ . Furthermore, assume that the triple  $(n, k, \mathbf{v}(t))$  satisfies the Condition 4 for  $(m_2, m_3)$  for each  $t \in [0, 1]$ . Then  $M_0 := \sup_{t \in [0, 1]} \{M_{\mathbf{v}(t), p, q}\} < \infty$  and  $\{\Gamma_{l, \mathbf{v}(t)}^{(R)}(x_i)\}_{i=1, \dots, \text{rank } p - \text{rank } q, l \in \mathbb{N}}$  satisfies the Condition 5. Here,  $l', m' \in \mathbb{N}$  of (ii), (iii) in Condition 5 can be taken  $l' = M_0$  and  $m' = m_2 + m_3$ .

**Proof.** The first condition (i) is clear from the definition of  $\Gamma_{l, \mathbf{v}(t)}^{(R)}$  and the fact that  $\mathbf{v}(t)$  is  $C^\infty$ . We recall the definitions of  $a_{\mathbf{v}}, c_{\mathbf{v}}, e_{\mathbf{v}}, \rho_{\mathbf{v}}, \varphi_{\mathbf{v}}, E_{\mathbf{v}}, F_{\mathbf{v}}, L_{\mathbf{v}}$  introduced in pp10–11 of Part I. As  $(n, k, p, q, \mathbf{v}(t))$  satisfies the Condition 2, from Lemma 2.9 of [O1], there exists a constant  $0 < s_{\mathbf{v}(t)} < 1$ , a state  $\varphi_{\mathbf{v}(t)}$ , and a positive element  $e_{\mathbf{v}(t)} \in M_{k+}$  and  $T_{\mathbf{v}(t)}$  satisfies the Spectral Property II with respect to  $(s_{\mathbf{v}(t)}, e_{\mathbf{v}(t)}, \varphi_{\mathbf{v}(t)})$ . Furthermore, we have  $s(e_{\mathbf{v}(t)}) = p$ ,  $s(\varphi_{\mathbf{v}(t)}) = q$ . In particular,  $[0, 1] \ni t \mapsto T_{\mathbf{v}(t)}$  is a  $C^\infty$ -map satisfying the (1), (2), (3) of Lemma B.3 with  $r_{T_{\mathbf{v}(t)}} = 1$ . Therefore, from Lemma B.3, there exists  $0 < s < 1$  such that  $\sigma(T_{\mathbf{v}(t)}) \setminus \{1\} \subset \mathcal{B}_s(0)$  for all  $t \in [0, 1]$ . Furthermore,  $[0, 1] \ni t \mapsto e_{\mathbf{v}(t)} = P_{\{1\}}^{T_{\mathbf{v}(t)}}(1)$  is a  $C^\infty$ -path of positive elements of  $M_k$  whose rank is equal to  $\text{rank } p$ . As  $a_{\mathbf{v}(t)} = d_{\mathbb{R}}(\sigma(e_{\mathbf{v}(t)}) \setminus \{0\}, \{0\})$ , we have  $a := \inf_{t \in [0, 1]} a_{\mathbf{v}(t)} > 0$  by Lemma A.6. Similarly,  $[0, 1] \ni t \mapsto \text{Tr} \left( P_{\{1\}}^{T_{\mathbf{v}(t)}}(\cdot) \right) \setminus \text{Tr } e_{\mathbf{v}(t)} = \varphi_{\mathbf{v}(t)} = \text{Tr } \rho_{\mathbf{v}(t)}(\cdot)$  is  $C^\infty$  and we

have  $c := \inf_{t \in [0,1]} c_{\mathbf{v}(t)} > 0$ . Therefore, for any  $s < s' < 1$  fixed, we have

$$C_1 := (ac)^{-1} k^2 \left( \sup_{t \in [0,1]} \sup_{|z|=s'} \|(z - T_{\mathbf{v}(t)})^{-1}\| \right) < \infty,$$

$$C_2 := \sup_{t \in [0,1]} F_{\mathbf{v}(t)} = \sup_{t \in [0,1]} \left( \sup_{N \in \mathbb{N}} \|T_{\mathbf{v}(t)}^N (1 - P_{\{1\}}^{T_{\mathbf{v}(t)}})\| + \|e_{\mathbf{v}(t)}\| \right) < \infty.$$

With these constants, we have

$$\sup_{t \in [0,1]} E_{\mathbf{v}(t)}(N) = \sup_{t \in [0,1]} (a_{\mathbf{v}(t)} c_{\mathbf{v}(t)})^{-1} k^2 \|T_{\mathbf{v}(t)}^N (1 - P_{\{1\}}^{T_{\mathbf{v}(t)}})\| \leq C_1(s')^N. \quad (6)$$

Fix some  $L_0 \in \mathbb{N}$  such that  $C_1(s')^{L_0} < \frac{1}{2}$ . Then we have

$$L_{\mathbf{v}(t)} = \inf \left\{ L \in \mathbb{N} \mid \sup_{N \geq L} E_{\mathbf{v}(t)}(N) < \frac{1}{2} \right\} \leq L_0, \quad t \in [0,1],$$

by (6). By Lemma 2.15 of Part I [O1], we have

$$M_0 := \sup_{t \in [0,1]} M_{\mathbf{v}(t), p, q} \leq \sup_{t \in [0,1]} L_{\mathbf{v}(t)} \leq L_0 < \infty.$$

This proves (ii) of *Condition 5* for  $l' = M_0$ .

(iii) of *Condition 5* for  $m' = m_2 + m_3$  follows from Lemma 2.14 of Part I [O1].

To prove (iv), note that

$$\sup_{t \in [0,1]} 2F_{\mathbf{v}(t)} E_{\mathbf{v}(t)}(m) (F_{\mathbf{v}(t)}^2 E_{\mathbf{v}(t)}(m) + 1) \leq 2C_1 C_2 (C_2^2 C_1 + 1) s'^m.$$

Choose  $L_1 \in \mathbb{N}$  such that  $8C_1 C_2 (C_2^2 C_1 + 1) \sqrt{m+1} s'^m < 1$  for any  $m \geq L_1$ . We use Lemma 2.17 of Part I [O1] replacing  $(l, m, r)$  in Lemma 2.17 of Part I [O1] by  $(N - l + 1, l - 1, 1)$  with  $l - 1 \geq \max\{m_1, L_0, L_1\} =: l''$ . Then we obtain (iv) with this  $l''$ .  $\square$

### 3 $C^1$ -classification of $\mathcal{H}(n, n_0, k_R, k_L)$

In this section, we show the following Proposition.

**Proposition 3.1.** *Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$  and  $k_R, k_L \in \mathbb{N} \cup \{0\}$ . Then for any  $\mathbb{B}_0, \mathbb{B}_1 \in \text{Class}(n, n_0, k_R, k_L)$ , and  $m_0, m_1 \in \mathbb{N}$  with  $m_0, m_1 \geq 2n_0^6(k_R + 1)(k_L + 1)$ , we have  $H_{\Phi_{m_0, \mathbb{B}_0}} \simeq_I H_{\Phi_{m_1, \mathbb{B}_1}}$ .*

First we check what the condition  $\lambda \in \text{Wo}'(k_R, k_L)$  implies.

**Lemma 3.2.** *Let  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , and  $(\lambda, \mathbb{D}, \mathbb{G}, Y) \in \mathcal{T}(k_R, k_L)$ . If furthermore  $\lambda \in \text{Wo}'(k_R, k_L)$ , then  $Y = 0$  and  $\mathbb{D}, \mathbb{G}$  satisfy the followings.:*

1. *For any  $1 \leq a_1, a_2 \leq k_R$ , either there exist  $\sigma(a_1, a_2) \in \{1, \dots, k_R\}$  and a nonzero  $\kappa(a_1, a_2) \in \mathbb{C}$  such that  $D_{a_1} D_{a_2} = \kappa(a_1, a_2) D_{\sigma(a_1, a_2)}$ , or  $D_{a_1} D_{a_2} = 0$ .*
2. *For any  $1 \leq b_1, b_2 \leq k_L$ , either there exist  $\sigma(b_1, b_2) \in \{1, \dots, k_L\}$  and a nonzero  $\kappa(b_1, b_2) \in \mathbb{C}$  such that  $G_{b_1} G_{b_2} = \kappa(b_1, b_2) G_{\sigma(b_1, b_2)}$ , or  $G_{b_1} G_{b_2} = 0$ .*

**Proof.** As  $Y$  belongs to  $\text{UT}_{0,k_R+k_L+1}$  and commutes with  $\Lambda_{\lambda}$  ((11) of [O1]), the assumption  $\lambda \in \text{Wo}'(k_R, k_L)$  implies that  $P_R^{(k_R, k_L)} Y P_R^{(k_R, k_L)} = P_L^{(k_R, k_L)} Y P_L^{(k_R, k_L)} = 0$ . As we also have  $P_R^{(k_R, k_L)} Y P_L^{(k_R, k_L)} = 0$ , we get  $Y = 0$ . To see the property of  $\mathbb{D}$  in the claim, let  $1 \leq a_1, a_2 \leq k_R$ . As the linear span of  $\{D_a\}_{a=1}^{k_R}$  is a subalgebra of  $\text{UT}_{0,k_R+1}$ ,  $D_{a_1} D_{a_2}$  can be expanded as  $D_{a_1} D_{a_2} = \sum_{a=1}^{k_R} c_a D_a$  with some coefficients  $c_a \in \mathbb{C}$ . From (9) of [O1], we have

$$\sum_{a=1}^{k_R} c_a \lambda_{-a} E_{-a0}^{(k_R, k_L)} = \Lambda_{\lambda} I_R^{(k_R, k_L)} (D_{a_1} D_{a_2}) \Lambda_{\lambda}^{-1} E_{00}^{(k_R, k_L)} = \lambda_{-a_1} \lambda_{-a_2} I_R^{(k_R, k_L)} (D_{a_1} D_{a_2}) E_{00}^{(k_R, k_L)} = \sum_{a=1}^{k_R} c_a \lambda_{-a_1} \lambda_{-a_2} E_{-a0}^{(k_R, k_L)}$$

From this, we have  $c_a = 0$  unless  $\lambda_{-a_1} \lambda_{-a_2} = \lambda_{-a}$ . As our  $\lambda$  is in  $\text{Wo}'(k_R, k_L)$ , such an  $a$  is at most one. Hence we obtain the claim. The assertion for  $\mathbb{G}$  is proven analogously.  $\square$

**Lemma 3.3.** *Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$  and  $\mathbb{B} \in \text{Class}(n, n_0, k_R, k_L)$ . Then  $(\lambda, \mathbb{D}, \mathbb{G}, 0) \in \mathcal{T}(k_R, k_L)$ ,  $\lambda \in \text{Wo}'(k_R, k_L)$  with respect to which  $\mathbb{B}$  belongs to  $\text{Class}(n, n_0, k_R, k_L)$  is uniquely determined. Furthermore, there exists a unique  $\omega \in M_{n_0}^{\times n}$  such that*

$$\lambda_i \omega_{\mu} \otimes E_{ii}^{(k_R, k_L)} = \left( \mathbb{I} \otimes E_{ii}^{(k_R, k_L)} \right) B_{\mu} \left( \mathbb{I} \otimes E_{ii}^{(k_R, k_L)} \right), \quad \mu = 1, \dots, n, \quad i = -k_R, \dots, k_L. \quad (7)$$

This  $\omega$  belongs to  $\text{Prim}_1(n, n_0)$ . (Recall Definition 1.6 [O1] for  $\text{Prim}_1(n, n_0)$ .)

**Definition 3.4.** We call the  $(\lambda, \mathbb{D}, \mathbb{G}, \omega)$  determined uniquely for  $\mathbb{B} \in \text{Class}(n, n_0, k_R, k_L)$  by Lemma 3.3, the quadruplet associated with  $\mathbb{B}$  and write it as  $(\lambda_{\mathbb{B}}, \mathbb{D}_{\mathbb{B}}, \mathbb{G}_{\mathbb{B}}, \omega_{\mathbb{B}})$ .

**Proof.** The formula (7) with  $i = 0$  corresponds to the formula in Lemma 3.2 [O1], and the latter Lemma implies  $\omega \in \text{Prim}_1(n, n_0)$ . As we have  $B_{\mu} \in M_{n_0} \otimes \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) \Lambda_{\lambda}$ , we obtain (7) for all  $i = -k_R, \dots, k_L$ . Suppose that  $\mathbb{B}$  is in  $\text{Class}(n, n_0, k_R, k_L)$  with respect to  $(\lambda, \mathbb{D}, \mathbb{G}, 0) \in \mathcal{T}(k_R, k_L)$  as well as  $(\lambda', \mathbb{D}', \mathbb{G}', 0) \in \mathcal{T}(k_R, k_L)$ . As  $r_{T_{\omega}} = 1 \neq 0$ , there exists a  $\mu$  such that  $\omega_{\mu} \neq 0$ . For this  $\mu$  and any  $i = -k_R, \dots, k_L$ , we have

$$\lambda_i \omega_{\mu} \otimes E_{ii}^{(k_R, k_L)} = \left( \mathbb{I} \otimes E_{ii}^{(k_R, k_L)} \right) B_{\mu} \left( \mathbb{I} \otimes E_{ii}^{(k_R, k_L)} \right) = \lambda'_i \omega_{\mu} \otimes E_{ii}^{(k_R, k_L)}, \quad (8)$$

and obtain  $\lambda = \lambda'$ . If  $k_R \in \mathbb{N}$ , then for any  $a \in \{1, \dots, k_R\}$  fixed, we have

$$\mathbb{I} \otimes I_R^{(k_R, k_L)} (D'_a) \Lambda_{\lambda}^l \in \mathcal{K}_l(\mathbb{B}) \hat{P}_R^{(n_0, k_R, k_L)} = M_{n_0} \otimes \text{span} \left\{ I_R^{(k_R, k_L)} (D_a) \Lambda_{\lambda}^l \right\}_{a=0}^{k_R}$$

for all  $l$  large enough. Here, we use a notation  $D_0 := P_R^{(k_R, 0)}$ . This means fixing  $l$  large enough, there exist  $\{Z_{il}\}_{i=0}^{k_R}$  such that

$$\mathbb{I} \otimes I_R^{(k_R, k_L)} (D'_a) \Lambda_{\lambda}^l = \sum_{i=0}^{k_R} Z_{il} \otimes I_R^{(k_R, k_L)} (D_i) \Lambda_{\lambda}^l.$$

As  $\{\lambda_{-i}\}_{i=0}^{k_R}$  is a set of  $k_R + 1$  distinct nonzero complex numbers, from Lemma C.7 Part I [O1], there exist  $\varsigma_i = (\varsigma_i(j))_{j=0}^{k_R} \in \mathbb{C}^{k_R+1}$ ,  $i = 0, \dots, k_R$  such that  $\sum_{j=0}^{k_R} \varsigma_i(j) \lambda_{-i'}^j = \delta_{i, i'}$ ,  $i, i' = 0, \dots, k_R$ . Using this  $\varsigma_i$ , we have

$$\begin{aligned} \mathbb{I} \otimes I_R^{(k_R, k_L)} (D'_a) \Lambda_{\lambda}^l &= \sum_{j=0}^{k_R} \varsigma_a(j) \lambda_{-a}^j \left( \mathbb{I} \otimes I_R^{(k_R, k_L)} (D'_a) \Lambda_{\lambda}^l \right) = \sum_{j=0}^{k_R} \varsigma_a(j) \left( \mathbb{I} \otimes \Lambda_{\lambda}^j \right) \left( \mathbb{I} \otimes I_R^{(k_R, k_L)} (D'_a) \Lambda_{\lambda}^l \right) \left( \mathbb{I} \otimes \Lambda_{\lambda}^{-j} \right) \\ &= \sum_{i=0}^{k_R} \sum_{j=0}^{k_R} \varsigma_a(j) \left( \mathbb{I} \otimes \Lambda_{\lambda}^j \right) \left( Z_{il} \otimes I_R^{(k_R, k_L)} (D_i) \Lambda_{\lambda}^l \right) \left( \mathbb{I} \otimes \Lambda_{\lambda}^{-j} \right) = \sum_{i=0}^{k_R} \sum_{j=0}^{k_R} \varsigma_a(j) \lambda_{-i}^j \left( Z_{il} \otimes I_R^{(k_R, k_L)} (D_i) \Lambda_{\lambda}^l \right) \\ &= Z_{al} \otimes I_R^{(k_R, k_L)} (D_a) \Lambda_{\lambda}^l. \end{aligned}$$



This means  $D'_a$  is proportional to  $D_a$ . By the normalization  $I_R^{(k_R, k_L)}(D_a)E_{00}^{(k_R, k_L)} = I_R^{(k_R, k_L)}(D'_a)E_{00}^{(k_R, k_L)} = E_{-a0}^{(k_R, k_L)}$ , we conclude  $D_a = D'_a$ . Similarly, we obtain  $G_b = G'_b$ .  $\square$

### 3.1 Constructive characterization of $\text{Class}(n, n_0, k_R, k_L)$

In order to classify  $\mathcal{H}(n, n_0, k_R, k_L)$ , we need to understand the property of  $\text{Class}(n, n_0, k_R, k_L)$ . We carry it out in this subsection. The number  $l_{\mathbb{B}}$  in (3) for  $\mathbb{B} \in \text{Class}(n, n_0, k_R, k_L)$  has a uniform upper bound.

**Lemma 3.5.** *Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$ . Let  $\mathbb{B} \in \text{Class}(n, n_0, k_R, k_L)$  with respect to  $(\lambda, \mathbb{D}, \mathbb{G}, 0)$ . Then  $l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, 0) \leq n_0^6(k_R + 1)(k_L + 1)$ .*

**Proof.** We would like to apply Lemma B.1 to see  $l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, 0) \leq n_0^6(k_R + 1)(k_L + 1)$ . The dimension of  $M_{n_0} \otimes \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G})\Lambda_{\lambda}^l$  is  $n_0^2(k_R + 1)(k_L + 1)$ . Therefore,  $\dim \mathcal{K}_l(\mathbb{B}) = \dim(M_{n_0} \otimes \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G})\Lambda_{\lambda}^l) = n_0^2(k_R + 1)(k_L + 1)$  for all  $l \geq l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, 0)$ .

On the other hand,  $\mathcal{K}_{l_{\omega}}(\mathbb{B})$  has an invertible element. (Recall Definition 1.6 of Part I [O1] for  $l_{\omega}$ .) To see this, note by the definition of  $l_{\omega}$ , that  $\mathbb{I}_{n_0} \in M_{n_0} = \mathcal{K}_{l_{\omega}}(\omega)$ . This means there exists a set of coefficients  $\{c_{\mu^{(l_{\omega})}}\}$  such that  $\sum_{\mu^{(l_{\omega})}} c_{\mu^{(l_{\omega})}} \widehat{\omega_{\mu^{(l_{\omega})}}}} = \mathbb{I}_{n_0}$ . Therefore, we have

$$\mathcal{K}_{l_{\omega}}(\mathbb{B}) \ni \sum_{\mu^{(l_{\omega})}} c_{\mu^{(l_{\omega})}} \widehat{B_{\mu^{(l_{\omega})}}} = \mathbb{I}_{n_0} \otimes \Lambda_{\lambda}^{l_{\omega}} + \text{an element of } M_{n_0} \otimes \text{UT}_{0, k_R + k_L + 1}.$$

From the right hand side, we see that this matrix is invertible.

Note that  $\dim \mathcal{K}_l(\mathbb{B}) = n_0^2(k_R + 1)(k_L + 1)$  if and only if  $\mathcal{K}_l(\mathbb{B}) = M_{n_0} \otimes \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G})\Lambda_{\lambda}^l$ , because we always have  $\mathcal{K}_l(\mathbb{B}) \subset M_{n_0} \otimes \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G})\Lambda_{\lambda}^l$ , due to the fact that  $\mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G})$  is an algebra invariant under operations  $\Lambda_{\lambda}^x(\cdot) \Lambda_{\lambda}^{-x}$ ,  $x \in \mathbb{Z}$ . Applying Lemma B.1, we obtain

$$l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, 0) \leq n_0^2(k_R + 1)(k_L + 1)l_{\omega} \leq n_0^6(k_R + 1)(k_L + 1).$$

For the second inequality, we used Lemma B.2.  $\square$

We would like to give a constructive characterization of  $\text{Class}(n, n_0, k_R, k_L)$ . To do so, we introduce the following notations.

**Definition 3.6.** Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $\lambda \in \mathbb{C}$ , and  $\omega \in \text{Prim}_1(n, n_0)$ . We define a linear subspace  $\mathcal{L}(\omega, \lambda)$  of  $\oplus_{\mu=1}^n M_{n_0}$  by

$$\mathcal{L}(\omega, \lambda) := \{(J\omega_{\mu} - \lambda\omega_{\mu}J)_{\mu=1}^n \mid J \in M_{n_0}\}.$$

We also define a linear map  $\Delta_{\omega, \lambda} : M_{n_0} \rightarrow \mathcal{L}(\omega, \lambda)$  by

$$\Delta_{\omega, \lambda}(J) := (J\omega_{\mu} - \lambda\omega_{\mu}J)_{\mu=1}^n, \quad J \in M_{n_0}.$$

**Definition 3.7.** Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$  and  $(\lambda, \mathbb{D}, \mathbb{G}, 0) \in \mathcal{T}(k_R, k_L)$  with  $\lambda \in \text{Wo}'(k_R, k_L)$ . We define

$$\mathfrak{H}_{\mathbb{D}}^R := \{1 \leq a \leq k_R \mid \text{there are no } 1 \leq a_1, a_2 \leq k_R \text{ such that } D_a \text{ is a scalar multiple of } D_{a_1} D_{a_2}\},$$

$$\mathfrak{H}_{\mathbb{G}}^L := \{1 \leq b \leq k_L \mid \text{there are no } 1 \leq b_1, b_2 \leq k_L \text{ such that } G_b \text{ is a scalar multiple of } G_{b_1} G_{b_2}\},$$

$$\text{GHR}(l, n_0, k_R, k_L, \mathbb{D}) := M_{n_0} \otimes \text{span}\{I_R^{(k_R, k_L)}(D_a)\Lambda_{\lambda}^l\}_{1 \leq a \notin \mathfrak{H}_{\mathbb{D}}^R},$$

$$\text{GHL}(l, n_0, k_R, k_L, \mathbb{G}) := M_{n_0} \otimes \text{span}\{I_L^{(k_R, k_L)}(G_b)\Lambda_{\lambda}^l\}_{1 \leq b \notin \mathfrak{H}_{\mathbb{G}}^L}.$$

**Lemma 3.8.** *Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $\lambda \in \mathbb{C}$  with  $\lambda \neq 1$ , and  $\omega \in \text{Prim}_1(n, n_0)$ . Then  $\Delta_{\omega, \lambda}$  is an injection.*

**Proof.** If  $J \in \ker \Delta_{\omega, \lambda}$ , then we have  $J\omega_\mu = \lambda\omega_\mu J$ , for all  $\mu = 1, \dots, n$ . From this, we obtain  $J\omega_{\mu^{(l)}} = \lambda^l \omega_{\mu^{(l)}} J$ , for any  $l \in \mathbb{N}$  and  $\mu^{(l)} \in \{1, \dots, n\}^{\times l}$ . As  $\omega \in \text{Prim}_1(n, n_0)$ , we have  $\mathcal{K}_l(\omega) = \mathbb{M}_{n_0}$  for  $l$  large enough. Hence, in particular, we have  $J = \lambda^l J$ , for  $l$  large enough. As  $\lambda \neq 1$ , this implies  $J = 0$ .  $\square$

**Definition 3.9.** Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$  and  $k_R, k_L \in \mathbb{N} \cup \{0\}$ . Let  $\mathbb{B} = (B_1, \dots, B_n)$  be an  $n$ -tuple of matrices in  $\mathbb{M}_{n_0} \otimes \text{UT}_{k_R+k_L+1} \subset \mathbb{M}_{n_0} \otimes \mathbb{M}_{k_R+k_L+1}$ . We say  $\mathbb{B}$  belongs to  $\text{Class}_1(n, n_0, k_R, k_L)$  if there exist  $(\lambda, \mathbb{D}, \mathbb{G}, 0) \in \mathcal{T}(k_R, k_L)$  with  $\lambda \in \text{Wo}'(k_R, k_L)$  and  $\omega \in \text{Prim}_1(n, n_0)$  satisfying the following conditions:

(1) For all  $i = -k_R, \dots, k_L$  and  $\mu = 1, \dots, n$ , we have

$$\lambda_i \omega_\mu \otimes E_{ii}^{(k_R, k_L)} = \left( \mathbb{I} \otimes E_{ii}^{(k_R, k_L)} \right) B_\mu \left( \mathbb{I} \otimes E_{ii}^{(k_R, k_L)} \right).$$

(2) For any  $\mu = 1, \dots, n$ , we have

$$\begin{aligned} B_\mu \hat{P}_R^{(n_0, k_R, k_L)} &= \omega_\mu \otimes \Lambda_\lambda P_R^{(k_R, k_L)} + \sum_{1 \leq a \leq k_R} \mathcal{X}_{\mu, a, \mathbb{B}} \otimes I_R^{(k_R, k_L)}(D_a) \Lambda_\lambda, \\ \hat{P}_L^{(n_0, k_R, k_L)} B_\mu &= \omega_\mu \otimes \Lambda_\lambda P_L^{(k_R, k_L)} + \sum_{1 \leq b \leq k_L} \mathcal{Y}_{\mu, b, \mathbb{B}} \otimes I_L^{(k_R, k_L)}(G_b) \Lambda_\lambda, \end{aligned} \quad (9)$$

where  $\mathbb{X}_{a, \mathbb{B}} := (\mathcal{X}_{\mu, a, \mathbb{B}})_{\mu=1}^n \notin \mathcal{L}(\omega, \lambda_{-a})$  and  $\mathbb{Y}_{b, \mathbb{B}} := (\mathcal{Y}_{\mu, b, \mathbb{B}})_{\mu=1}^n \notin \mathcal{L}(\omega, \lambda_b^{-1})$ , for any  $a \in \mathfrak{H}_{\mathbb{D}}^R$  and  $b \in \mathfrak{H}_{\mathbb{G}}^L$ .

*Remark 3.10.* We say  $\mathbb{B}$  belongs to  $\text{Class}_1(n, n_0, k_R, k_L)$  with respect to  $(\lambda, \mathbb{D}, \mathbb{G}, \omega)$ , when we would like to state explicitly. We denote  $\mathcal{X}_{\mathbb{B}} := \{\mathcal{X}_{\mu, a, \mathbb{B}}\}_{a, \mu}$ ,  $\mathcal{Y}_{\mathbb{B}} := \{\mathcal{Y}_{\mu, b, \mathbb{B}}\}_{b, \mu}$ , and call them the sets of matrices associated with  $\mathbb{B}$ . (Note that they are determined uniquely by  $\mathbb{B}$  due to the independence of  $\{\mathbb{I}\} \cup \{I_R^{(k_R, k_L)}(D_a)\}_{a=1}^{k_R} \cup \{I_L^{(k_R, k_L)}(G_b)\}_{b=1}^{k_L} \cup \{E_{-a, b}^{(k_R, k_L)}\}_{a=1, \dots, k_R, b=1, \dots, k_L}$ .)

The rest of this subsection is devoted to showing the following Lemma:

**Lemma 3.11.** *Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$  and  $k_R, k_L \in \mathbb{N} \cup \{0\}$ . Let  $(\lambda, \mathbb{D}, \mathbb{G}, 0) \in \mathcal{T}(k_R, k_L)$  with  $\lambda \in \text{Wo}'(k_R, k_L)$  and  $\omega \in \text{Prim}_1(n, n_0)$ . Then  $\mathbb{B} \in (\mathbb{M}_{n_0} \otimes \text{UT}_{k_R+k_L+1})^{\times n}$  belongs to  $\text{Class}(n, n_0, k_R, k_L)$  with associated quadruple  $(\lambda, \mathbb{D}, \mathbb{G}, \omega)$  if and only if it belongs to  $\text{Class}_1(n, n_0, k_R, k_L)$  with respect to  $(\lambda, \mathbb{D}, \mathbb{G}, \omega)$ .*

*Remark 3.12.* The advantage of this characterization is that the conditions for  $\text{Class}_1(n, n_0, k_R, k_L)$  is easier to check, when we construct a path  $\mathbb{B}(t)$ .

**Lemma 3.13.** *Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $k_R, k_L \in \mathbb{N}$  and  $(\lambda, \mathbb{D}, \mathbb{G}, 0) \in \mathcal{T}(k_R, k_L)$  with  $\lambda \in \text{Wo}'(k_R, k_L)$ . For any  $b \in \{1, \dots, k_L\}$ , there exists an  $m \in \mathbb{N}$  with  $m \leq k_L$ ,  $b_1, \dots, b_m \in \mathfrak{H}_{\mathbb{G}}^L$ , and  $c_b \in \mathbb{C}$  such that  $G_b = c_b G_{b_1} \cdots G_{b_m}$ .*

**Proof.** Set  $n_b := \min\{j - i : (G_b)_{ij} \neq 0\}$ . By the definition, we have  $1 \leq n_b \leq k_L$ . The claim is trivial when  $b \in \mathfrak{H}_{\mathbb{G}}^L$ . If  $b \notin \mathfrak{H}_{\mathbb{G}}^L$ , then there exist  $b_1, b_2 \in \{1, \dots, k_L\}$  such that  $\sigma(b_1, b_2) = b$ , and  $G_b = \kappa(b_1, b_2)^{-1} G_{b_1} G_{b_2}$  with nonzero  $\kappa(b_1, b_2) \in \mathbb{C}$ . From this, we see  $2 \leq n_{b_1} + n_{b_2} \leq n_b$ . If furthermore  $b_1 \notin \mathfrak{H}_{\mathbb{G}}^L$ , we repeat the same argument to obtain  $b'_{11}, b'_{12} \in \{1, \dots, k_L\}$  such that  $G_{b_1} \propto G_{b'_{11}} G_{b'_{12}} G_{b_2}$ . We repeat this procedure and obtain  $b_1, b_2, \dots, b_m$  such that  $G_b \propto G_{b_1} G_{b_2} \cdots G_{b_m}$  and  $m \leq n_{b_1} + n_{b_2} + \cdots + n_{b_m} \leq n_b$ . If some of  $b_i$  is not in  $\mathfrak{H}_{\mathbb{G}}^L$ , we repeat the same argument to split  $b_i$  into two. However, this procedure stops in finite time because of the bound  $m \leq n_b \leq k_L$ . Namely, at some point, all of  $b_1, \dots, b_m$  will be in  $\mathfrak{H}_{\mathbb{G}}^L$ .  $\square$

**Lemma 3.14.** *Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$ . Let  $\mathbb{B} \in \text{Class}_1(n, n_0, k_R, k_L)$  with respect to  $(\lambda, \mathbb{D}, \mathbb{G}, \omega)$ . Then we have  $M_{n_0} \otimes P_L^{(k_R, k_L)} \Lambda_\lambda^l \subset \hat{P}_L^{(n_0, k_R, k_L)} \mathcal{K}_l(\mathbb{B})$  (resp.  $M_{n_0} \otimes P_R^{(k_R, k_L)} \Lambda_\lambda^l \subset \mathcal{K}_l(\mathbb{B}) \hat{P}_R^{(n_0, k_R, k_L)}$ ), for  $l$  large enough. Furthermore, if  $k_L \in \mathbb{N}$ , (resp.  $k_R \in \mathbb{N}$ ), for any  $b \in \{1, \dots, k_L\}$  (resp.  $a \in \{1, \dots, k_R\}$ ), we have  $M_{n_0} \otimes I_L^{(k_R, k_L)}(G_b) \Lambda_\lambda^l \subset \hat{P}_L^{(n_0, k_R, k_L)} \mathcal{K}_l(\mathbb{B})$  (resp.  $M_{n_0} \otimes I_R^{(k_R, k_L)}(D_a) \Lambda_\lambda^l \subset \mathcal{K}_l(\mathbb{B}) \hat{P}_R^{(n_0, k_R, k_L)}$ ), for  $l$  large enough.*

**Proof.** Note that the first statement is trivial if  $k_L = 0$  (resp.  $k_R = 0$ ), from the primitivity of  $\omega$ . We prove the Lemma for  $k_L \in \mathbb{N}$ ,  $b \in \{1, \dots, k_L\}$ . The proof for  $k_R \in \mathbb{N}$ ,  $a \in \{1, \dots, k_R\}$  is the same. Assume  $k_L \in \mathbb{N}$ . For each  $b \in \{1, \dots, k_L\}$ , we define

$$\check{l}_b := \inf \left\{ l \in \mathbb{N} \mid M_{n_0} \otimes I_L^{(k_R, k_L)}(G_b) \Lambda_\lambda^{l'} \subset \hat{P}_L^{(n_0, k_R, k_L)} \mathcal{K}_{l'}(\mathbb{B}) + \text{GHL}(l', n_0, k_R, k_L, \mathbb{G}), \text{ for all } l' \geq l \right\}.$$

First we see  $\check{l}_b < \infty$  for all  $b \in \mathfrak{H}_\mathbb{G}^L$ . From (9) and  $(\lambda, \mathbb{D}, \mathbb{G}, 0) \in \mathcal{T}(k_R, k_L)$ , we can check inductively that for any  $l \in \mathbb{N}$  and  $\mu^{(l)} \in \{1, \dots, n\}^{\times l}$ ,  $\hat{P}_L^{(n_0, k_R, k_L)} \widehat{B_{\mu^{(l)}}}$  is of the form

$$\hat{P}_L^{(n_0, k_R, k_L)} \widehat{B_{\mu^{(l)}}} = \widehat{\omega_{\mu^{(l)}}} \otimes \Lambda_\lambda^l P_L^{(k_R, k_L)} + \sum_{b=1}^{k_L} \mathcal{Z}_{\mu^{(l)}, b} \otimes I_L^{(k_R, k_L)}(G_b) \Lambda_\lambda^l,$$

with  $\mathcal{Z}_{\mu^{(l)}, b} \in M_{n_0}$ .

Recall that we have  $M_{n_0} = \mathcal{K}_l(\omega)$  for all  $l \geq l_\omega$ . Therefore, for any  $l \geq l_\omega$  and  $\alpha, \beta \in \{1, \dots, n_0\}$ , there exists  $X_{\alpha, \beta}^{(l)} \in \mathcal{K}_l(\mathbb{B})$  such that  $\hat{P}_L^{(n_0, k_R, k_L)} X_{\alpha, \beta}^{(l)}$  is the of the form

$$\hat{P}_L^{(n_0, k_R, k_L)} X_{\alpha, \beta}^{(l)} = e_{\alpha, \beta}^{(n_0)} \otimes \Lambda_\lambda^l P_L^{(k_R, k_L)} + \sum_{b=1}^{k_L} \mathcal{W}_{\alpha, \beta, b}^{(l)} \otimes I_L^{(k_R, k_L)}(G_b) \Lambda_\lambda^l,$$

with some  $\mathcal{W}_{\alpha, \beta, b}^{(l)} \in M_{n_0}$ .

For any  $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \{1, \dots, n_0\}$  and  $l_1, l_2 \geq l_\omega$ , we have

$$\begin{aligned} \hat{P}_L^{(n_0, k_R, k_L)} \mathcal{K}_{l_1+l_2}(\mathbb{B}) &\ni \hat{P}_L^{(n_0, k_R, k_L)} X_{\alpha_1, \beta_1}^{(l_1)} X_{\alpha_2, \beta_2}^{(l_2)} = \hat{P}_L^{(n_0, k_R, k_L)} X_{\alpha_1, \beta_1}^{(l_1)} \hat{P}_L^{(n_0, k_R, k_L)} X_{\alpha_2, \beta_2}^{(l_2)} \\ &= \delta_{\beta_1, \alpha_2} e_{\alpha_1, \beta_2}^{(n_0)} \otimes \Lambda_\lambda^{l_1+l_2} P_L^{(k_R, k_L)} + \sum_{b=1}^{k_L} \left( \mathcal{W}_{\alpha_1, \beta_1, b}^{(l_1)} e_{\alpha_2, \beta_2}^{(n_0)} + \lambda_b^{-l_1} e_{\alpha_1, \beta_1}^{(n_0)} \mathcal{W}_{\alpha_2, \beta_2, b}^{(l_2)} \right) \otimes I_L^{(k_R, k_L)}(G_b) \Lambda_\lambda^{(l_1+l_2)} \\ &\quad + \text{an element of } \text{GHL}(l_1 + l_2, n_0, k_R, k_L, \mathbb{G}). \end{aligned} \tag{10}$$

We use the following Lemma which can be proven by the same argument as Lemma 7.14 of [O2].

**Lemma 3.15.** *Let  $n, n_0 \in \mathbb{N}$ ,  $k_R \in \mathbb{N} \cup \{0\}$  and  $k_L \in \mathbb{N}$ . Let  $\mathbb{B} \in \text{Class}_1(n, n_0, k_R, k_L)$  with respect to  $(\lambda, \mathbb{D}, \mathbb{G}, \omega)$ . Suppose that there exist an  $l' \in \mathbb{N}$  and matrices  $y_b \in M_{n_0}$ ,  $b = 1, \dots, k_L$ , such that*

$$\sum_{b=1}^{k_L} y_b \otimes I_L^{(k_R, k_L)}(G_b) \Lambda_\lambda^{l'} \in \hat{P}_L^{(n_0, k_R, k_L)} \mathcal{K}_{l'}(\mathbb{B}).$$

Then for any  $b \in \mathfrak{H}_\mathbb{G}^L$  with  $y_b \neq 0$ , we have

$$M_{n_0} \otimes I_L^{(k_R, k_L)}(G_b) \Lambda_\lambda^l \subset \hat{P}_L^{(n_0, k_R, k_L)} \mathcal{K}_l(\mathbb{B}) + \text{GHL}(l, n_0, k_R, k_L, \mathbb{G}),$$

for all  $l \geq 2l_\omega + k_L + l' - 1$ .

Suppose that  $\tilde{l}_b = \infty$  for some  $b \in \mathfrak{H}_{\mathbb{G}}^L$ . Applying Lemma 3.15 to (10), we see that  $x_{\alpha,\beta}^{(l)} := \mathcal{W}_{\alpha,\beta,b}^{(l)}$  and  $\lambda := \lambda_b$  satisfy the conditions in Lemma 6.6 of Part II [O2] with  $l_0 = l_{\omega}$ . By the latter Lemma, there exists  $J \in \mathbf{M}_{n_0}$  such that  $\mathcal{W}_{\alpha,\beta,b}^{(l)} = J e_{\alpha,\beta}^{(n_0)} - \lambda_b^{-l} e_{\alpha\beta}^{(n_0)} J$ ,  $l \geq l_{\omega}$ ,  $\alpha, \beta = 1, \dots, n_0$ ,  $l \geq l_{\omega}$ .

Substituting this and (9), we have

$$\begin{aligned} \hat{P}_L^{(n_0, k_R, k_L)} \mathcal{K}_{l_1+l_2+1}(\mathbb{B}) &\ni \sum_{\alpha_1, \alpha_2=1}^{n_0} \left( \hat{P}_L^{(n_0, k_R, k_L)} X_{\alpha_1, \alpha_1}^{(l_1)} B_{\mu} X_{\alpha_2, \alpha_2}^{(l_2)} - \left\langle \chi_{\alpha_1}^{(n_0)}, \omega_{\mu} \chi_{\alpha_2}^{(n_0)} \right\rangle X_{\alpha_1, \alpha_2}^{(l_1+l_2+1)} \right) \\ &= \lambda_b^{-l_1} (-J \omega_{\mu} + \mathcal{Y}_{\mu, b, \mathbb{B}} + \lambda_b^{-1} \omega_{\mu} J) \otimes I_L^{(k_R, k_L)}(G_b) \Lambda_{\lambda}^{l_1+l_2+1} \\ &\quad + \text{elements of } \mathbf{M}_{n_0} \otimes \text{span}\{I_L^{(k_R, k_L)}(G_{b'}) \Lambda_{\lambda}^{l_1+l_2+1}\}_{b' \neq b, b'=1, \dots, k_L} \end{aligned}$$

Applying Lemma 3.15 and from the assumption  $\tilde{l}_b = \infty$ , we get

$$\mathcal{Y}_{\mu, b, \mathbb{B}} = J \omega_{\mu} - \lambda_b^{-1} \omega_{\mu} J, \quad \mu = 1, \dots, n.$$

This means  $\mathbb{Y}_{b, \mathbb{B}} := (\mathcal{Y}_{\mu, b, \mathbb{B}})_{\mu=1}^n \in \mathcal{L}(\omega, \lambda_b^{-1})$ , which contradict the assumption  $\mathbb{B} \in \text{Class}_1(n, n_0, k_R, k_L)$ . Hence we have  $\tilde{l}_b < \infty$  for all  $b \in \mathfrak{H}_{\mathbb{G}}^L$ .

Let  $\tilde{l} := \max\{\tilde{l}_b \mid b \in \mathfrak{H}_{\mathbb{G}}^L\} < \infty$ . Then for each  $b \in \mathfrak{H}_{\mathbb{G}}^L$ ,  $\alpha, \beta \in \{1, \dots, n_0\}$  and  $l \geq \tilde{l}$ , there exists  $C_{l, \alpha, \beta, b} \in \text{GHL}(l, n_0, k_R, k_L, \mathbb{G})$  such that  $e_{\alpha, \beta}^{(n_0)} \otimes I_L^{(k_R, k_L)}(G_b) \Lambda_{\lambda}^l + C_{l, \alpha, \beta, b} \in \hat{P}_L^{(n_0, k_R, k_L)} \mathcal{K}_l(\mathbb{B})$ . For each  $i \in \{1, \dots, k_L\}$ , we set

$$m_i := \max\{m \in \mathbb{N} \mid \text{there exist } b_1, \dots, b_m \in \mathfrak{H}_{\mathbb{G}}^L \text{ such that } G_i \propto G_{b_1} \dots G_{b_m}\}.$$

From the proof of Lemma 3.13, we see that  $m_i \leq k_L$ .

Let us consider  $i \in \{1, \dots, k_L\}$ , with a decomposition  $G_i = c_i G_{b_1} \dots G_{b_{m_i}}$ ,  $b_1, \dots, b_{m_i} \in \mathfrak{H}_{\mathbb{G}}^L$ . Then for all  $l_1, \dots, l_{m_i} \geq \tilde{l}$  and  $\alpha, \beta \in \{1, \dots, n_0\}$ , we have

$$\begin{aligned} &\left( e_{\alpha, \beta}^{(n_0)} \otimes I_L^{(k_R, k_L)}(G_{b_1}) \Lambda_{\lambda}^{l_1} + C_{l_1, \alpha, \beta, b_1} \right) \left( e_{\beta, \beta}^{(n_0)} \otimes I_L^{(k_R, k_L)}(G_{b_2}) \Lambda_{\lambda}^{l_2} + C_{l_2, \beta, \beta, b_2} \right) \dots \left( e_{\beta, \beta}^{(n_0)} \otimes I_L^{(k_R, k_L)}(G_{b_{m_i}}) \Lambda_{\lambda}^{l_{m_i}} + C_{l_{m_i}, \beta, \beta, b_{m_i}} \right) \\ &\in \hat{P}_L^{(n_0, k_R, k_L)} \mathcal{K}_{l_1+l_2+\dots+l_{m_i}}(\mathbb{B}). \end{aligned}$$

From this, we see that for all  $i \in \{1, \dots, k_L\}$ ,  $l \geq m_i \tilde{l}$  and  $\alpha, \beta \in \{1, \dots, n_0\}$ , we have

$$e_{\alpha, \beta}^{(n_0)} \otimes I_L^{(k_R, k_L)}(G_i) \Lambda_{\lambda}^l + \text{an element of } \mathbf{M}_{n_0} \otimes \text{span}\{I_L^{(k_R, k_L)}(G_{b'}) \Lambda_{\lambda}^l\}_{b' \in \{1, \dots, k_L\}, m_{b'} > m_i} \in \hat{P}_L^{(n_0, k_R, k_L)} \mathcal{K}_l(\mathbb{B}). \quad (11)$$

In particular, if there is no  $b'$  such that  $m_{b'} > m_i$ , then  $\mathbf{M}_{n_0} \otimes I_L^{(k_R, k_L)}(G_i) \Lambda_{\lambda}^l \subset \hat{P}_L^{(n_0, k_R, k_L)} \mathcal{K}_l(\mathbb{B})$  for  $l$  large enough. By induction in  $m_i$  (from large  $m_i$  to small ones), we prove the second part of the Lemma. By this, for any  $\alpha, \beta \in \{1, \dots, n_0\}$ , we have

$$e_{\alpha, \beta}^{(n_0)} \otimes \Lambda_{\lambda}^l P_L^{(k_R, k_L)} = \hat{P}_L^{(n_0, k_R, k_L)} X_{\alpha, \beta}^{(l)} - \sum_{b=1}^{k_L} \mathcal{W}_{\alpha, \beta, b}^{(l)} \otimes I_L^{(k_R, k_L)}(G_b) \Lambda_{\lambda}^l \in \hat{P}_L^{(n_0, k_R, k_L)} \mathcal{K}_l(\mathbb{B}), \quad (12)$$

for  $l$  large enough.  $\square$

Now we give a proof of Lemma 3.11

**Proof of Lemma 3.11.** Suppose that  $\mathbb{B}$  belongs to  $\text{Class}_1(n, n_0, k_R, k_L)$  with respect to  $(\lambda, \mathbb{D}, \mathbb{G}, \omega)$ . We would like to show that  $\mathbb{B}$  belongs to  $\text{Class}(n, n_0, k_R, k_L)$  with associated quadruple  $(\lambda, \mathbb{D}, \mathbb{G}, \omega)$ . By Lemma 3.14 and Definition 3.9, there exists  $l'_{\mathbb{B}} \in \mathbb{N}$  such that

$$\begin{aligned} \mathcal{K}_l(\mathbb{B}) \hat{P}_R^{(n_0, k_R, k_L)} &= \mathbf{M}_{n_0} \otimes \text{span}\left\{I_R^{(k_R, k_L)}(D_a) \Lambda_{\lambda}^l\right\}_{a=0}^{k_R}, \\ \hat{P}_D^{(n_0, k_R, k_L)} \mathcal{K}_l(\mathbb{B}) &= \mathbf{M}_{n_0} \otimes \text{span}\left\{I_L^{(k_R, k_L)}(G_b) \Lambda_{\lambda}^l\right\}_{b=0}^{k_L}. \end{aligned} \quad (13)$$

for all  $l \geq l'_\mathbb{B}$ . Here we denoted  $D_0 := P_R^{(k_R, 0)}$ ,  $G_0 := P_L^{(0, k_L)}$ . Note that this corresponds to the property 1. of Lemma 7.2 Part II [O2]. The only difference here is that our  $B_\mu$ s may have term in  $\widehat{P}_L^{(n_0, k_R, k_L)}(M_{n_0} \otimes M_{k_R + k_L + 1}) \widehat{P}_R^{(n_0, k_R, k_L)}$ . However the existence of such terms does not affect the argument in Section 7 of Part II [O2] and statements of Lemma 7.3, 7.6, 7.7 can be proven. This implies that  $\mathbb{B}$  is in  $\text{Class}(n, n_0, k_R, k_L)$  with the associated quadruple  $(\lambda, \mathbb{D}, \mathbb{G}, \omega)$ .

Next let us prove  $\text{Class}(n, n_0, k_R, k_L) \subset \text{Class}_1(n, n_0, k_R, k_L)$ . Assume that  $\mathbb{B}$  belongs to  $\text{Class}(n, n_0, k_R, k_L)$  with the associated quadruple  $(\lambda, \mathbb{D}, \mathbb{G}, \omega)$ . We would like to show that  $\mathbb{B}$  belongs to  $\text{Class}_1(n, n_0, k_R, k_L)$  with respect to  $(\lambda, \mathbb{D}, \mathbb{G}, \omega)$ . (1) of Definition 3.9 corresponds to Lemma 3.3 and Definition 3.4. We have to show (2) of Definition 3.9. Note that  $\widehat{P}_L^{(n_0, k_R, k_L)} B_\mu$  is of the form

$$\widehat{P}_L^{(n_0, k_R, k_L)} B_\mu = \omega_\mu \otimes \Lambda_\lambda P_L^{(k_R, k_L)} + \sum_{1 \leq b' \leq k_L} \mathcal{Y}_{\mu, b', \mathbb{B}} \otimes I_L^{(k_R, k_L)}(G_{b'}) \Lambda_\lambda,$$

for any  $\mu = 1, \dots, n$ . Suppose that  $\mathbb{Y}_{b, \mathbb{B}} = (\mathcal{Y}_{\mu, b, \mathbb{B}})_{\mu=1}^n \in \mathcal{L}(\omega, \lambda_b^{-1})$  for some  $b \in \mathfrak{H}_\mathbb{G}^L$ ,  $\mu = 1, \dots, n$ . Then there exists  $J_b$  such that  $\mathcal{Y}_{\mu, b, \mathbb{B}} = J_b \omega_\mu - \lambda_b^{-1} \omega_\mu J_b$ . We can check inductively that for any  $l \in \mathbb{N}$  and  $\mu^{(l)} \in \{1, \dots, n\}^{\times l}$ ,  $\widehat{P}_L^{(n_0, k_R, k_L)} \widehat{B}_{\mu^{(l)}}$  is of the form

$$\widehat{P}_L^{(n_0, k_R, k_L)} \widehat{B}_{\mu^{(l)}} = \widehat{\omega}_{\mu^{(l)}} \otimes \Lambda_\lambda^l P_L^{(k_R, k_L)} + \sum_{b'=1}^{k_L} \mathcal{Z}_{\mu^{(l)}, b'} \otimes I_L^{(k_R, k_L)}(G_{b'}) \Lambda_\lambda^l,$$

with  $\mathcal{Z}_{\mu^{(l)}, b'} \in M_{n_0}$  and in particular, we have

$$\mathcal{Z}_{\mu^{(l)}, b} = J_b \widehat{\omega}_{\mu^{(l)}} - \lambda_b^{-l} \widehat{\omega}_{\mu^{(l)}} J_b.$$

Now, as  $\mathbb{B} \in \text{Class}(n, n_0, k_R, k_L)$ , for  $l \in \mathbb{N}$  large enough, there exist coefficients  $\{\alpha_{\mu^{(l)}}\}_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}}$  such that

$$\begin{aligned} \mathbb{I} \otimes I_L^{(k_R, k_L)}(G_b) \Lambda_\lambda^l &= \widehat{P}_L^{(n_0, k_R, k_L)} \sum_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \alpha_{\mu^{(l)}} \widehat{B}_{\mu^{(l)}} \\ &= \sum_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \alpha_{\mu^{(l)}} \widehat{\omega}_{\mu^{(l)}} \otimes \Lambda_\lambda^l P_L^{(k_R, k_L)} + \sum_{b'=1}^{k_L} \sum_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \alpha_{\mu^{(l)}} \mathcal{Z}_{\mu^{(l)}, b'} \otimes I_L^{(k_R, k_L)}(G_{b'}) \Lambda_\lambda^l. \end{aligned}$$

From the linearly independence of  $\{G_b\}_{b=1}^{k_L}$ , we obtain

$$\sum_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \alpha_{\mu^{(l)}} \widehat{\omega}_{\mu^{(l)}} = 0, \quad \mathbb{I} = \sum_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \alpha_{\mu^{(l)}} \mathcal{Z}_{\mu^{(l)}, b} = \sum_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \alpha_{\mu^{(l)}} (J_b \widehat{\omega}_{\mu^{(l)}} - \lambda_b^{-l} \widehat{\omega}_{\mu^{(l)}} J_b) = 0.$$

This is a contradiction. Therefore, we have  $\mathbb{Y}_{b, \mathbb{B}} \notin \mathcal{L}(\omega, \lambda_b^{-1})$  for all  $b \in \mathfrak{H}_\mathbb{G}^L$ . Similarly, we have  $\mathbb{X}_{a, \mathbb{B}} \notin \mathcal{L}(\omega, \lambda_{-a})$  for all  $a \in \mathfrak{H}_\mathbb{D}^R$ . Hence the condition (2) of Definition 3.9 holds.  $\square$

## 3.2 Proof of Proposition 3.1

In this subsection, we prove Proposition 3.1. For this, we prove the following Lemma.

**Lemma 3.16.** *Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$  and  $k_R, k_L \in \mathbb{N} \cup \{0\}$ . Then for any  $\mathbb{B}_0, \mathbb{B}_1 \in \text{Class}(n, n_0, k_R, k_L)$ , there exists a continuous and piecewise  $C^\infty$ -map  $\mathbb{B} : [0, 1] \rightarrow (M_{n_0} \otimes M_{k_R + k_L + 1})^{\times n}$  such that  $\mathbb{B}(0) = \mathbb{B}_0$ ,  $\mathbb{B}(1) = \mathbb{B}_1$  and  $\mathbb{B}(t) \in \text{Class}(n, n_0, k_R, k_L)$  for all  $t \in [0, 1]$ .*

First we connect each element in  $\text{Class}(n, n_0, k_R, k_L)$  to an element in  $\text{Class}_2(n, n_0, k_R, k_L)$ , defined as follows.

**Definition 3.17.** Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ , and  $k_R, k_L \in \mathbb{N} \cup \{0\}$ . We say  $\mathbb{B} \in (\text{M}_{n_0} \otimes \text{M}_{k_R+k_L+1})^{\times n}$  belongs to  $\text{Class}_2(n, n_0, k_R, k_L)$  if  $\mathbb{B} \in \text{Class}_1(n, n_0, k_R, k_L) (= \text{Class}(n, n_0, k_R, k_L))$  with respect to some  $(\lambda, \mathbb{D}, \mathbb{G}, \omega)$ , and the associated set of matrices  $\{\mathcal{X}_{\mu,a,\mathbb{B}}\}_{a,\mu}, \{\mathcal{Y}_{\mu,b,\mathbb{B}}\}_{b,\mu}$  (see Remark 3.10) satisfy  $\mathbb{X}_{a,\mathbb{B}} = (\mathcal{X}_{\mu,a,\mathbb{B}})_{\mu=1}^n \notin \mathcal{L}(\omega, \lambda_{-a}), \mathbb{Y}_{b,\mathbb{B}} = (\mathcal{Y}_{\mu,b,\mathbb{B}})_{\mu=1}^n \notin \mathcal{L}(\omega, \lambda_b^{-1})$  for any  $1 \leq a \leq k_R$  and  $1 \leq b \leq k_L$ .

**Lemma 3.18.** Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , and  $\mathbb{B}_0 \in \text{Class}(n, n_0, k_R, k_L)$ . Then there exist  $\mathbb{B}_1 \in \text{Class}_2(n, n_0, k_R, k_L)$  and a  $C^\infty$ -map  $\mathbb{B} : [0, 1] \rightarrow (\text{M}_{n_0} \otimes \text{M}_{k_R+k_L+1})^{\times n}$  such that  $\mathbb{B}(0) = \mathbb{B}_0$ ,  $\mathbb{B}(1) = \mathbb{B}_1$  and  $\mathbb{B}(t) \in \text{Class}(n, n_0, k_R, k_L)$  for all  $t \in [0, 1]$ .

**Proof.** Let  $(\lambda_0, \mathbb{D}_0, \mathbb{G}_0, \omega_0)$  be the quadruplet associated with  $\mathbb{B}_0 \in \text{Class}(n, n_0, k_R, k_L)$ . Let  $\{\mathcal{X}_{0,\mu,a}\}, \{\mathcal{Y}_{0,\mu,b}\}$ , be the set of matrices associated with  $\mathbb{B}_0$ . Set  $\mathcal{X}_{1,\mu,a} := \mathcal{X}_{0,\mu,a}$  and  $\mathcal{Y}_{1,\mu,b} := \mathcal{Y}_{0,\mu,b}$  for each  $a \in \mathfrak{H}_{\mathbb{D}_0}^R$  and  $b \in \mathfrak{H}_{\mathbb{G}_0}^L$ . As  $\mathbb{B}_0 \in \text{Class}_1(n, n_0, k_R, k_L)$ , by Lemma 3.11, we have  $(\mathcal{X}_{0,\mu,a})_\mu = (\mathcal{X}_{1,\mu,a})_\mu \notin \mathcal{L}(\omega_0, \lambda_{-a})$  for  $a \in \mathfrak{H}_{\mathbb{D}_0}^R$ , and  $(\mathcal{Y}_{0,\mu,b})_\mu = (\mathcal{Y}_{1,\mu,b})_\mu \notin \mathcal{L}(\omega_0, \lambda_b^{-1})$ , for  $b \in \mathfrak{H}_{\mathbb{G}_0}^L$ . Note that for any  $\lambda \in \mathbb{C}$ ,  $\dim \mathcal{L}(\omega_0, \lambda) \leq n_0^2 < \dim(\bigoplus_{\mu=1}^n \text{M}_{n_0}) = n_0^2 n$ , as  $n \geq 2$ . Therefore, for each  $a \in \{1, \dots, k_R\} \setminus \mathfrak{H}_{\mathbb{D}_0}^R$  and  $b \in \{1, \dots, k_L\} \setminus \mathfrak{H}_{\mathbb{G}_0}^L$ , we can find  $(\mathcal{X}_{1,\mu,a})_\mu \in \left(\bigoplus_{\mu=1}^n \text{M}_{n_0}\right) \setminus \mathcal{L}(\omega_0, \lambda_{-a})$ ,  $(\mathcal{Y}_{1,\mu,b})_\mu \in \left(\bigoplus_{\mu=1}^n \text{M}_{n_0}\right) \setminus \mathcal{L}(\omega_0, \lambda_b^{-1})$ .

For each  $t \in [0, 1]$ , define  $\mathbb{B}(t) \in (\text{M}_{n_0} \otimes \text{M}_{k_R+k_L+1})^{\times n}$  by

$$\begin{aligned} B_\mu(t) &:= \omega_{0,\mu} \otimes \Lambda_{\lambda_0} + \sum_{a=1}^{k_R} ((1-t)\mathcal{X}_{0,\mu,a} + t\mathcal{X}_{1,\mu,a}) \otimes I_R^{(k_R, k_L)}(D_{0,a}) \Lambda_{\lambda_0} \\ &+ \sum_{b=1}^{k_L} ((1-t)\mathcal{Y}_{0,\mu,b} + t\mathcal{Y}_{1,\mu,b}) \otimes I_L^{(k_R, k_L)}(G_{0,b}) \Lambda_{\lambda_0} + \overline{\hat{P}_L^{(n_0, k_R, k_L)}} B_{0,\mu} \overline{\hat{P}_R^{(n_0, k_R, k_L)}}, \quad \mu = 1, \dots, n. \end{aligned}$$

Clearly,  $\mathbb{B} : [0, 1] \rightarrow (\text{M}_{n_0} \otimes \text{M}_{k_R+k_L+1})^{\times n}$  is a  $C^\infty$ -map such that  $\mathbb{B}(0) = \mathbb{B}_0$ . It is easy to check for all  $t \in [0, 1]$  that  $\mathbb{B}(t) \in \text{Class}_1(n, n_0, k_R, k_L) = \text{Class}(n, n_0, k_R, k_L)$  such that

$$\begin{aligned} \lambda_{\mathbb{B}(t)} &= \lambda_0, \mathbb{D}_{\mathbb{B}(t)} = \mathbb{D}_0, \mathbb{G}_{\mathbb{B}(t)} = \mathbb{G}_0, \omega_{\mathbb{B}(t)} = \omega_0, \\ \mathcal{X}_{\mu,a,\mathbb{B}(t)} &= (1-t)\mathcal{X}_{0,\mu,a} + t\mathcal{X}_{1,\mu,a}, \mathcal{Y}_{\mu,b,\mathbb{B}(t)} = (1-t)\mathcal{Y}_{0,\mu,b} + t\mathcal{Y}_{1,\mu,b}. \end{aligned}$$

As we have  $(\mathcal{X}_{1,\mu,a})_\mu \notin \mathcal{L}(\omega_0, \lambda_{-a}), (\mathcal{Y}_{1,\mu,b})_\mu \notin \mathcal{L}(\omega_0, \lambda_b^{-1}), \mathbb{B}_1 := \mathbb{B}(1)$  belongs to  $\text{Class}_2(n, n_0, k_R, k_L)$ .  $\square$

Next we consider elements in  $\text{Class}_2(n, n_0, k_R, k_L)$  given as follows.

**Lemma 3.19.** Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , and  $\mathbb{B} \in \text{Class}_2(n, n_0, k_R, k_L)$ . Let  $(\lambda, \mathbb{D}, \mathbb{G}, \omega)$  be the quadruplet associated with  $\mathbb{B}$ . Let  $\mathcal{X}, \mathcal{Y}$  be the set of matrices associated with  $\mathbb{B}$ . Define  $\tilde{\mathbb{B}} = (\tilde{B}_1, \dots, \tilde{B}_n) \in (\text{M}_{n_0} \otimes \text{M}_{k_R+k_L+1})^{\times n}$  by

$$\tilde{B}_\mu = \omega_\mu \otimes \Lambda_\lambda + \sum_{a=1}^{k_R} \mathcal{X}_{\mu,a} \otimes E_{-a,0}^{(k_R, k_L)} \Lambda_\lambda + \sum_{b=1}^{k_L} \mathcal{Y}_{\mu,b} \otimes E_{0,b}^{(k_R, k_L)} \Lambda_\lambda, \quad \mu = 1, \dots, n.$$

Then  $\tilde{\mathbb{B}} \in \text{Class}_2(n, n_0, k_R, k_L)$ , with

$$\begin{aligned} \lambda_{\tilde{\mathbb{B}}} &= \lambda, \mathbb{D}_{\tilde{\mathbb{B}}} = (E_{-1,0}^{(k_R, 0)}, \dots, E_{-k_R,0}^{(k_R, 0)}), \mathbb{G}_{\tilde{\mathbb{B}}} = (E_{0,1}^{(0, k_L)}, \dots, E_{0,k_L}^{(0, k_L)}), \omega_{\tilde{\mathbb{B}}} = \omega, \\ \mathcal{X}_{\tilde{\mathbb{B}}} &= \mathcal{X}, \mathcal{Y}_{\tilde{\mathbb{B}}} = \mathcal{Y}. \end{aligned}$$

**Proof.** As  $\mathbb{B} \in \text{Class}_2(n, n_0, k_R, k_L)$ , we have  $\mathcal{X}_{\mu,a} \notin \mathcal{L}(\omega, \lambda_{-a})$ ,  $\mathcal{Y}_{\mu,b} \notin \mathcal{L}(\omega, \lambda_b^{-1})$ . With this observation, it is easy to check that  $\tilde{\mathbb{B}} \in \text{Class}_2(n, n_0, k_R, k_L)$ .  $\square$

**Definition 3.20.** For  $\mathbb{B} \in \text{Class}_2(n, n_0, k_R, k_L)$ , we denote the  $\tilde{\mathbb{B}}$  given in Lemma 3.19 by  $\mathbb{S}_{\mathbb{B}} := \tilde{\mathbb{B}}$ .

Next we connect  $\mathbb{B}_0 \in \text{Class}_2(n, n_0, k_R, k_L)$  to  $\mathbb{S}_{\mathbb{B}_0}$ .

**Lemma 3.21.** Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , and  $\mathbb{B}_0 \in \text{Class}_2(n, n_0, k_R, k_L)$ . Set  $\mathbb{B}_1 := \mathbb{S}_{\mathbb{B}_0}$ . Then there exists a  $C^\infty$ -map  $\mathbb{B} : [0, 1] \rightarrow (\text{M}_{n_0} \otimes \text{M}_{k_R+k_L+1})^{\times n}$  such that  $\mathbb{B}(0) = \mathbb{B}_0$ ,  $\mathbb{B}(1) = \mathbb{B}_1$  and  $\mathbb{B}(t) \in \text{Class}_2(n, n_0, k_R, k_L)$  for all  $t \in [0, 1]$ .

**Proof.** Let  $(\lambda, \mathbb{D}, \mathbb{G}, \omega)$  be the quadruplet associated with  $\mathbb{B}_0$ . Let  $\mathcal{X}, \mathcal{Y}$  be the set of matrices associated with  $\mathbb{B}_0$ .

Define for each  $t \in [0, 1]$ ,  $1 \leq a \leq k_R$ , and  $1 \leq b \leq k_L$ ,

$$D_a(t) := E_{-a,0}^{(k_R,0)} + (1-t)D_a Q_{R,-1}^{(k_R,0)}, \quad G_b(t) := E_{0,b}^{(0,k_L)} + (1-t)Q_{L,1}^{(0,k_L)} G_b.$$

We consider  $\mathbb{D}(t) := (D_1(t), \dots, D_{k_R}(t))$  and  $\mathbb{G}(t) := (G_1(t), \dots, G_{k_L}(t))$ .

We have  $\mathbb{D}(0) = \mathbb{D}$  and  $\mathbb{G}(0) = \mathbb{G}$ . We also have  $\mathbb{D}(1) = (E_{-1,0}^{(k_R,0)}, E_{-2,0}^{(k_R,0)}, \dots, E_{-k_R,0}^{(k_R,0)})$  and  $\mathbb{G}(1) = (E_{0,1}^{(0,k_L)}, E_{0,2}^{(0,k_L)}, \dots, E_{0,k_L}^{(0,k_L)})$ . Furthermore, we have

$$\begin{aligned} \Lambda_\lambda I_L^{(k_R, k_L)}(G_b(t)) &= \Lambda_\lambda \left( E_{0,b}^{(k_R, k_L)} + (1-t)Q_{L,1}^{(k_R, k_L)} I_L^{(k_R, k_L)}(G_b) \right) \\ &= \lambda_b^{-1} \left( E_{0,b}^{(k_R, k_L)} + (1-t)Q_{L,1}^{(k_R, k_L)} I_L^{(k_R, k_L)}(G_b) \right) \Lambda_\lambda = \lambda_b^{-1} I_L^{(k_R, k_L)}(G_b(t)) \Lambda_\lambda, \end{aligned}$$

for  $1 \leq b \leq k_L$  and  $t \in [0, 1]$ . Similarly, we have  $\Lambda_\lambda I_R^{(k_R, k_L)}(D_a(t)) = \lambda_{-a} I_R^{(k_R, k_L)}(D_a(t)) \Lambda_\lambda$  for  $1 \leq a \leq k_R$  and  $t \in [0, 1]$ .

We claim  $\mathbb{D}(t) := (D_1(t), \dots, D_{k_R}(t)) \in \mathcal{C}^R(k_R)$ , and  $\mathbb{G}(t) := (G_1(t), \dots, G_{k_L}(t)) \in \mathcal{C}^L(k_L)$ , for each  $t \in [0, 1]$ . We check  $\mathbb{G}(t) := (G_1(t), \dots, G_{k_L}(t)) \in \mathcal{C}^L(k_L)$ . That  $\mathbb{D}(t) \in \mathcal{C}^R(k_R)$  can be checked in the same way. It is clear from the definition that  $E_{00}^{(0,k_L)} G_b(t) = E_{0b}^{(0,k_L)}$  and  $G_b(t) \in \text{UT}_{0, k_L+1}$ .

We have to check the condition 2. of Definition 1.7 Part I [O1]. Take arbitrary  $1 \leq b_1, b_2 \leq k_L$ . Note from Lemma 3.3 [O1] that

$$\begin{aligned} G_{b_1} G_{b_2} &= \left( E_{0,b_1}^{(0,k_L)} + Q_{L,1}^{(0,k_L)} G_{b_1} \right) \left( E_{0,b_2}^{(0,k_L)} + Q_{L,1}^{(0,k_L)} G_{b_2} \right) \\ &= E_{0,b_1}^{(0,k_L)} G_{b_2} + Q_{L,1}^{(0,k_L)} G_{b_1} G_{b_2}. \end{aligned} \tag{14}$$

Recall Lemma 3.2. Suppose that there exist  $\sigma(b_1, b_2) \in \{1, \dots, k_L\}$  and nonzero  $\kappa(b_1, b_2) \in \mathbb{C}$  such that

$$G_{b_1} G_{b_2} = \kappa(b_1, b_2) G_{\sigma(b_1, b_2)}. \tag{15}$$

The equations(14), (15), imply

$$\kappa(b_1, b_2) E_{0, \sigma(b_1, b_2)}^{(0,k_L)} = \kappa(b_1, b_2) E_{0,0}^{(0,k_L)} G_{\sigma(b_1, b_2)} = E_{0,0}^{(0,k_L)} G_{b_1} G_{b_2} = E_{0,b_1}^{(0,k_L)} G_{b_2}, \tag{16}$$

and

$$\kappa(b_1, b_2) Q_{L,1}^{(0,k_L)} G_{\sigma(b_1, b_2)} = Q_{L,1}^{(0,k_L)} G_{b_1} G_{b_2}. \tag{17}$$

Substituting (16) and (17), we have

$$\begin{aligned} G_{b_1}(t) G_{b_2}(t) &= \left( E_{0,b_1}^{(0,k_L)} + (1-t)Q_{L,1}^{(0,k_L)} G_{b_1} \right) \left( E_{0,b_2}^{(0,k_L)} + (1-t)Q_{L,1}^{(0,k_L)} G_{b_2} \right) \\ &= (1-t)E_{0,b_1}^{(0,k_L)} G_{b_2} + (1-t)^2 Q_{L,1}^{(0,k_L)} G_{b_1} G_{b_2} = (1-t) \left( \kappa(b_1, b_2) E_{0, \sigma(b_1, b_2)}^{(0,k_L)} + (1-t) \kappa(b_1, b_2) Q_{L,1}^{(0,k_L)} G_{\sigma(b_1, b_2)} \right) \\ &= (1-t) \kappa(b_1, b_2) G_{\sigma(b_1, b_2)}(t). \end{aligned}$$

Hence we have  $G_{b_1}(t)G_{b_2}(t) \in \text{span}\{G_b(t)\}_{b=1}^{k_L}$ ,  $t \in [0, 1]$ , when  $G_{b_1}G_{b_2} \neq 0$ .

Suppose  $G_{b_1}G_{b_2} = 0$ . Then from (14), we have

$$E_{0,b_1}^{(0,k_L)}G_{b_2} = 0, \quad Q_{L,1}^{(0,k_L)}G_{b_1}G_{b_2} = 0.$$

Substituting this, we have

$$G_{b_1}(t)G_{b_2}(t) = \left(E_{0,b_1}^{(0,k_L)} + (1-t)Q_{L,1}^{(0,k_L)}G_{b_1}\right) \left(E_{0,b_2}^{(0,k_L)} + (1-t)Q_{L,1}^{(0,k_L)}G_{b_2}\right) = 0.$$

This proves  $G_{b_1}(t)G_{b_2}(t) \in \text{span}\{G_b(t)\}_{b=1}^{k_L}$ ,  $t \in [0, 1]$ , when  $G_{b_1}G_{b_2} = 0$ . Hence we obtain  $\mathbb{G}(t) \in \mathcal{C}^L(k_L)$ .

Now we define the path  $\mathbb{B}(t)$ . We set  $\mathbb{B}(t) := (B_1(t), \dots, B_n(t))$  with

$$B_\mu(t) := \omega_\mu \otimes \Lambda_\lambda + \sum_{a=1}^{k_R} \mathcal{X}_{\mu,a} \otimes I_R^{(k_R,k_L)}(D_a(t)) \Lambda_\lambda + \sum_{b=1}^{k_L} \mathcal{Y}_{\mu,b} \otimes I_L^{(k_R,k_L)}(G_b(t)) \Lambda_\lambda + (1-t) \overline{\hat{P}_L^{(n_0,k_R,k_L)}} B_\mu \overline{\hat{P}_R^{(n_0,k_R,k_L)}},$$

for  $\mu = 1, \dots, n$  and  $t \in [0, 1]$ . It is trivial that  $\mathbb{B} : [0, 1] \rightarrow (\mathbb{M}_{n_0} \otimes \mathbb{M}_{k_R+k_L+1})^{\times n}$  is  $C^\infty$ . Furthermore, we have  $\mathbb{B}(0) = \mathbb{B}_0$ , and  $\mathbb{B}(1) = \mathbb{S}(\mathbb{B}) = \mathbb{B}_1$ .

For all  $t \in [0, 1]$ , it is easy to check  $\mathbb{B}(t) \in \text{Class}_2(n, n_0, k_R, k_L)$  with  $\lambda_{\mathbb{B}(t)} = \lambda$ ,  $\mathbb{D}_{\mathbb{B}(t)} = \mathbb{D}(t)$ ,  $\mathbb{G}_{\mathbb{B}(t)} = \mathbb{G}(t)$ ,  $\omega_{\mathbb{B}(t)} = \omega$ ,  $\mathcal{X}_{\mathbb{B}(t)} = \mathcal{X}$ , and  $\mathcal{Y}_{\mathbb{B}(t)} = \mathcal{Y}$ .  $\square$

Now we connect elements of  $\text{Class}_2(n, n_0, k_R, k_L)$ .

**Lemma 3.22.** *Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$  and  $k_R, k_L \in \mathbb{N} \cup \{0\}$ . Then for any  $\mathbb{B}_0, \mathbb{B}_1 \in \text{Class}_2(n, n_0, k_R, k_L)$ , there exists a continuous and piecewise  $C^\infty$ -map  $\mathbb{B} : [0, 1] \rightarrow (\mathbb{M}_{n_0} \otimes \mathbb{M}_{k_R+k_L+1})^{\times n}$  such that  $\mathbb{B}(0) = \mathbb{B}_0$ ,  $\mathbb{B}(1) = \mathbb{B}_1$  and  $\mathbb{B}(t) \in \text{Class}_2(n, n_0, k_R, k_L)$  for all  $t \in [0, 1]$ .*

**Proof.** Let  $(\lambda_i, \mathbb{D}_i, \mathbb{G}_i, \omega_i)$  be the quadruplet associated with  $\mathbb{B}_i$ , and  $\mathcal{X}_i, \mathcal{Y}_i$  the set of matrices associated with  $\mathbb{B}_i$ , for  $i = 0, 1$ .

From [BO], Theorem 5.1, there exists a continuous and piecewise  $C^\infty$ -map  $\tilde{\omega} : [0, 1] \rightarrow (\mathbb{M}_{n_0})^{\times n}$  such that  $\tilde{\omega}(0) = \omega_0$ ,  $\tilde{\omega}(1) = \omega_1$  and  $\tilde{\omega}(t) \in \text{Prim}(n, n_0)$ . (Note that the proof there provides continuous and piecewise  $C^\infty$ -path.) By Appendix D of [BO], we see that  $T_{\tilde{\omega}(t)}$  satisfies the conditions of Lemma B.3. In particular, we have  $r_{T_{\tilde{\omega}(t)}} > 0$  and  $\omega(t) := r_{T_{\tilde{\omega}(t)}}^{-\frac{1}{2}} \tilde{\omega}(t)$  belongs to  $\text{Prim}_1(n, n_0)$ . By Lemma B.3, the path  $\omega : [0, 1] \rightarrow (\mathbb{M}_{n_0})^{\times n}$  is continuous and piecewise  $C^\infty$ , and  $\omega(0) = \omega_0$ ,  $\omega(1) = \omega_1$ .

Furthermore, there exists a continuous and piecewise  $C^\infty$ -path  $\lambda : [0, 1] \rightarrow \mathbb{C}^{k_R+k_L+1}$  with  $\lambda(0) = \lambda_0$ ,  $\lambda(1) = \lambda_1$  such that  $\lambda(t) \in \text{Wo}'(k_R, k_L)$  for all  $t \in [0, 1]$ . To see this, choose  $\mathbf{r} := (r_j)_{j=-k_R}^{k_L} \in \mathbb{C}^{k_R+k_L+1}$  such that  $r_0 = 1$ ,  $0 < r_{-k_R} < \dots < r_{-1} < 1$ , and  $0 < r_{k_L} < \dots < r_1 < 1$ . Decompose each  $\lambda_{j,i} \in \mathbb{C}$  as  $\lambda_{j,i} = e^{i\theta_{j,i}} |\lambda_{j,i}|$ ,  $\theta_{j,i} \in [0, 2\pi)$ , for each  $i = 0, 1$  and  $j = -k_R, \dots, k_L$ . Define  $\lambda(t) := (\lambda_j(t))_{j=-k_R}^{k_L}$ ,  $t \in [0, 1]$  by

$$\lambda_j(t) := \begin{cases} e^{i\theta_{j,0}} ((1-3t)|\lambda_{j,0}| + 3tr_j), & t \in \left[0, \frac{1}{3}\right], \\ e^{i((-3t+2)\theta_{j,0} + (3t-1)\theta_{j,1})} r_j, & t \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ e^{i\theta_{j,1}} ((3t-2)|\lambda_{j,1}| + 3(1-t)r_j), & t \in \left[\frac{2}{3}, 1\right]. \end{cases} \quad (18)$$

It is straight forward to check that  $\lambda(t) := (\lambda_{-k_R}(t), \dots, \lambda_{k_L}(t))$  satisfies the claimed properties.



Next we consider paths of vectors in  $\bigoplus_{\mu=1}^n M_{n_0}$ . For each  $\alpha, \beta \in n_0, a = 1, \dots, k_R, b = 1, \dots, k_L$  and  $t \in [0, 1]$ , define

$$\zeta_{\alpha, \beta, a}^R(t) := \Delta_{\omega(t), \lambda_{-a}(t)} \left( e_{\alpha, \beta}^{(n_0)} \right), \quad \zeta_{\alpha, \beta, b}^L(t) := \Delta_{\omega(t), \lambda_b(t)^{-1}} \left( e_{\alpha, \beta}^{(n_0)} \right).$$

These define continuous and piecewise  $C^\infty$ -paths  $\zeta_{\alpha, \beta, a}^R, \zeta_{\alpha, \beta, b}^L$  in  $\bigoplus_{\mu=1}^n M_{n_0}$ . By Lemma 3.8, for each  $a = 1, \dots, k_R$  (resp.  $b = 1, \dots, k_L$ ) and  $t \in [0, 1]$ ,  $\{\zeta_{\alpha, \beta, a}^R(t)\}_{\alpha, \beta}$  (resp.  $\{\zeta_{\alpha, \beta, b}^L(t)\}_{\alpha, \beta}$ ) are linearly independent. Regarding  $\bigoplus_{\mu=1}^n M_{n_0}$  a Hilbert space with inner product  $\langle \bigoplus_{\mu} a_{\mu}, \bigoplus_{\mu} b_{\mu} \rangle = \sum_{\mu} \text{Tr } a_{\mu}^* b_{\mu}$ , let  $P_a(t)$  be the orthogonal projection onto  $\text{span}\{\zeta_{\alpha, \beta, a}^R(t)\}_{\alpha, \beta}$  for each  $a = 1, \dots, k_R$ .

We have  $(1 - P_a(0))(\mathcal{X}_{0, \mu, a})_{\mu=1}^n \neq 0$  and  $(1 - P_a(1))(\mathcal{X}_{1, \mu, a})_{\mu=1}^n \neq 0$  for all  $a = 1, \dots, k_R$  because  $\mathbb{B}_0, \mathbb{B}_1 \in \text{Class}_2(n, n_0, k_R, k_L)$ . Applying Lemma A.5 to  $\text{span}\{\zeta_{\alpha, \beta, a}^R(t)\}_{\alpha, \beta}$ , we obtain a continuous and piecewise  $C^\infty$ -path  $\mathbb{X}_a : [0, 1] \rightarrow \bigoplus_{\mu=1}^n M_{n_0}$  such that  $(1 - P_a(t))\mathbb{X}_a(t) \neq 0$  for all  $t \in [0, 1]$  and  $\mathbb{X}_a(0) = (\mathcal{X}_{0, \mu, a})_{\mu=1}^n, \mathbb{X}_a(1) = (\mathcal{X}_{1, \mu, a})_{\mu=1}^n$  for each  $a = 1, \dots, k_R$ . In other words, we obtain a continuous and piecewise  $C^\infty$ -path  $[0, 1] \ni t \mapsto \mathbb{X}_a(t) = (\mathcal{X}_{\mu, a}(t))_{\mu=1}^n \in \left( \bigoplus_{\mu=1}^n M_{n_0} \right) \setminus \mathcal{L}(\omega(t), \lambda_a(t))$  such that  $\mathbb{X}_a(0) = \mathbb{X}_{0, a}$  and  $\mathbb{X}_a(1) = \mathbb{X}_{1, a}$  for each  $a = 1, \dots, k_R$ . Similarly we obtain a continuous and piecewise  $C^\infty$ -path  $[0, 1] \ni t \mapsto \mathbb{Y}_a(t) = (\mathcal{Y}_{\mu, b}(t))_{\mu=1}^n \in \left( \bigoplus_{\mu=1}^n M_{n_0} \right) \setminus \mathcal{L}(\omega(t), \lambda_b^{-1}(t))$  such that  $\mathbb{Y}_b(0) = \mathbb{Y}_{0, b}$  and  $\mathbb{Y}_b(1) = \mathbb{Y}_{1, b}$  for each  $b = 1, \dots, k_L$ .

Now we define the path  $\mathbb{B}(t)$ . We set  $\mathbb{B}(t) := (B_1(t), \dots, B_n(t))$  with

$$B_{\mu}(t) := \omega_{\mu}(t) \otimes \Lambda_{\lambda(t)}^{(k_R, k_L)} + \sum_{a=1}^{k_R} \mathcal{X}_{\mu, a}(t) \otimes E_{-a, 0}^{(k_R, k_L)} \Lambda_{\lambda(t)}^{(k_R, k_L)} + \sum_{b=1}^{k_L} \mathcal{Y}_{\mu, b}(t) \otimes E_{0, b}^{(k_R, k_L)} \Lambda_{\lambda(t)}^{(k_R, k_L)}$$

for  $\mu = 1, \dots, n$  and  $t \in [0, 1]$ .

It is trivial that  $\mathbb{B} : [0, 1] \rightarrow (M_{n_0} \otimes M_{k_R + k_L + 1})^{\times n}$  is continuous and piecewise  $C^\infty$ . Furthermore, we have  $\mathbb{B}(0) = \mathbb{S}_{\mathbb{B}_0}$  and  $\mathbb{B}(1) = \mathbb{S}_{\mathbb{B}_1}$ . For all  $t \in [0, 1]$ , it is easy to check  $\mathbb{B}(t) \in \text{Class}_2(n, n_0, k_R, k_L)$  with  $\lambda_{\mathbb{B}(t)} = \lambda(t), \omega_{\mathbb{B}(t)} = \omega(t), \mathbb{D}_{\mathbb{B}(t)} = (E_{-a, 0}^{(k_R, 0)})_{a=1}^{k_R}, \mathbb{G}_{\mathbb{B}(t)} = (E_{0, b}^{(0, k_L)})_{b=1}^{k_L}, \mathcal{X}_{\mathbb{B}(t)} = \mathcal{X}(t)$ , and  $\mathcal{Y}_{\mathbb{B}(t)} = \mathcal{Y}(t)$ .  $\square$

**Proof of Lemma 3.16.** For  $\mathbb{A}_0, \mathbb{A}_1 \in \text{Class}(n, n_0, k_R, k_L)$ , we denote  $\mathbb{A}_0 \approx \mathbb{A}_1$  if there exists a continuous and piecewise  $C^\infty$ -map  $\mathbb{A} : [0, 1] \rightarrow (M_{n_0} \otimes M_{k_R + k_L + 1})^{\times n}$  such that  $\mathbb{A}(0) = \mathbb{A}_0, \mathbb{A}(1) = \mathbb{A}_1$  and  $\mathbb{A}(t) \in \text{Class}(n, n_0, k_R, k_L)$  for all  $t \in [0, 1]$ . Let  $\mathbb{B}_0, \mathbb{B}_1 \in \text{Class}(n, n_0, k_R, k_L)$ . By Lemma 3.18, there exists  $\mathbb{B}'_0, \mathbb{B}'_1 \in \text{Class}_2(n, n_0, k_R, k_L)$  such that  $\mathbb{B}_0 \approx \mathbb{B}'_0, \mathbb{B}_1 \approx \mathbb{B}'_1$ . Lemma 3.21 implies  $\mathbb{B}'_0 \approx \mathbb{S}_{\mathbb{B}'_0}, \mathbb{B}'_1 \approx \mathbb{S}_{\mathbb{B}'_1}$ . Finally, Lemma 3.22 implies  $\mathbb{S}_{\mathbb{B}'_0} \approx \mathbb{S}_{\mathbb{B}'_1}$ , proving Lemma 3.16.  $\square$

**Proof of Proposition 3.1.** Let  $\mathbb{B}_0, \mathbb{B}_1 \in \text{Class}(n, n_0, k_R, k_L)$ . Let  $2n_0^6(k_R + 1)(k_L + 1) \leq m_0, m_1 \in \mathbb{N}$ . By Lemma 3.16, there exists a continuous and piecewise  $C^\infty$ -path  $\mathbb{B} : [0, 1] \rightarrow (M_{n_0} \otimes M_{k_R + k_L + 1})^{\times n}$  such that  $\mathbb{B}(0) = \mathbb{B}_0, \mathbb{B}(1) = \mathbb{B}_1$  and  $\mathbb{B}(t) \in \text{Class}(n, n_0, k_R, k_L), t \in [0, 1]$ . From Lemma 3.6 Part I [O1],  $(n, n_0(k_L + k_R + 1), \hat{P}_R^{(n_0, k_R, k_L)}, \hat{P}_L^{(n_0, k_R, k_L)}, \mathbb{B}(t))$  satisfies *Condition 2*. By Lemma 3.7 Part I [O1],  $(n, n_0(k_L + k_R + 1), \hat{P}_R^{(n_0, k_R, k_L)}, \hat{P}_L^{(n_0, k_R, k_L)}, \mathbb{B}(t))$  satisfies *Condition 3* for  $l_{\mathbb{B}(t)} = l_{\mathbb{B}(t)}(n, n_0, k_R, k_L, \lambda(t), \mathbb{D}(t), \mathbb{G}(t), 0)$ . By Lemma 3.8 of [O1],  $(n, n_0(k_L + k_R + 1), \mathbb{B}(t))$  satisfies *Condition 4* for  $(l_{\mathbb{B}(t)}, l_{\mathbb{B}(t)})$ . These facts and the estimate  $l_{\mathbb{B}(t)} \leq n_0^6(k_R + 1)(k_L + 1)$  from Lemma 3.5 guarantee the condition of Proposition 2.4 with  $m_1 = m_2 = m_3 = n_0^6(k_R + 1)(k_L + 1)$ . From Proposition 2.4, the paths  $[0, 1] \ni t \rightarrow \Gamma_{m, \mathbb{B}(t)}^{(R)} \left( e_{\alpha, \beta}^{(n_0)} \otimes E_{-a, b}^{(k_R, k_L)} \right)$  satisfies *Condition 5*. From Lemma 2.3, this implies Proposition 3.1.  $\square$

## 4 Classification of $\mathcal{H}(n)$

In this section, we prove Theorem 1.8. Given Proposition 3.1, it suffices to find a concrete  $C^1$ -path of gapped Hamiltonians connecting  $H_{\Phi_{m_1, \mathbb{B}_1}}$  with  $\mathbb{B}_1 \in \text{Class}(n, n_0, k_R, k_L)$  and  $H_{\Phi_{m_2, \mathbb{B}_2}}$  with  $\mathbb{B}_2 \in \text{Class}(n, 1, n_0(k_R + 1) - 1, n_0(k_L + 1) - 1)$ .

### 4.1 The path with singularity

We introduce several notations. Throughout Section 4, we fix numbers  $0 < \kappa < 1$  and  $\theta \in 2\pi(\mathbb{R} \setminus \mathbb{Q})$ . Let  $2 \leq n_0 \in \mathbb{N}$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$ . Define vectors  $\mathbf{r} = (r_\alpha)_{\alpha=1}^{n_0} \in \mathbb{C}^{n_0}$ ,  $\boldsymbol{\lambda}_L = (\lambda_{L,\alpha})_{\alpha=-(n_0-1)}^{n_0-1} \in \mathbb{C}^{2n_0-1}$ , and  $\boldsymbol{\lambda}_R = (\lambda_{R,i})_{i=-k_R}^{k_L} \in \mathbb{C}^{k_R+k_L+1}$  by

$$\begin{aligned} r_\alpha &:= (\kappa e^{i\theta})^{\alpha-1}, \quad \text{for } \alpha = 1, \dots, n_0, \\ \lambda_{L,\alpha} &:= (\kappa e^{i\theta})^{|\alpha|}, \quad \text{for } \alpha = -(n_0-1), \dots, -1, 0, 1, \dots, (n_0-1), \\ \lambda_{R,j} &:= (\kappa e^{i\theta})^{|j|n_0}, \quad \text{for } j = -k_R, \dots, -1, 0, 1, \dots, k_L. \end{aligned} \quad (19)$$

Note that we have  $\boldsymbol{\lambda}_L \in \text{Wo}'(n_0-1, n_0-1)$  and  $\boldsymbol{\lambda}_R \in \text{Wo}'(k_R, k_L)$ . We also define diagonal matrices

$$R := \sum_{\alpha=1}^{n_0} r_\alpha e_{\alpha,\alpha}^{(n_0)} \in \text{M}_{n_0}, \quad \Lambda_{\boldsymbol{\lambda}_L} := \sum_{\alpha=-(n_0-1)}^{n_0-1} \lambda_{L,\alpha} E_{\alpha,\alpha}^{(n_0-1, n_0-1)} \in \text{M}_{2n_0-1}, \quad \Lambda_{\boldsymbol{\lambda}_R} := \sum_{i=-k_R}^{k_L} \lambda_{R,i} E_{ii}^{(k_R, k_L)} \in \text{M}_{k_R+k_L+1}.$$

We define vectors

$$\begin{aligned} \eta_0^{(n_0)} &:= \sum_{\alpha=2}^{n_0} \chi_\alpha^{(n_0)} \in \mathbb{C}^{n_0}, \\ \eta_R^{(n_0-1, n_0-1)} &:= \sum_{\alpha=-(n_0-1)}^{-1} f_\alpha^{(n_0-1, n_0-1)}, \quad \eta_L^{(n_0-1, n_0-1)} := \sum_{\beta=1}^{n_0-1} f_\beta^{(n_0-1, n_0-1)} \in \mathbb{C}^{2n_0-1}, \\ \eta_R^{(k_R, k_L)} &:= \sum_{i=-k_R}^{-1} f_i^{(k_R, k_L)}, \quad \eta_L^{(k_R, k_L)} := \sum_{i=1}^{k_L} f_i^{(k_R, k_L)} \in \mathbb{C}^{k_R+k_L+1}. \end{aligned}$$

(Recall the definition of  $f_i^{(k_R, k_L)}$  from Appendix A of [O1].)

From these vectors, we define

$$V := \left| \eta_R^{(k_R, k_L)} \right\rangle \left\langle f_0^{(k_R, k_L)} \right| + \left| f_0^{(k_R, k_L)} \right\rangle \left\langle \eta_L^{(k_R, k_L)} \right| \in \text{M}_{k_R+k_L+1}, \quad (20)$$

and

$$\begin{aligned} K_1^{(L)} &:= \left| \chi_1^{(n_0)} \right\rangle \left\langle \eta_0^{(n_0)} \right|, \quad K_1^{(R)} := \Lambda_{\boldsymbol{\lambda}_R}, \quad \tilde{K}_1^{(L)} := \left| f_0^{(n_0-1, n_0-1)} \right\rangle \left\langle \eta_L^{(n_0-1, n_0-1)} \right|, \\ K_2^{(L)} &:= R, \quad K_2^{(R)} := V \Lambda_{\boldsymbol{\lambda}_R}, \quad \tilde{K}_2^{(L)} := \Lambda_{\boldsymbol{\lambda}_L}, \\ K_3^{(L)} &:= \left| \eta_0^{(n_0)} \right\rangle \left\langle \chi_1^{(n_0)} \right|, \quad K_3^{(R)} := \Lambda_{\boldsymbol{\lambda}_R}, \quad \tilde{K}_3^{(L)} := \left| \eta_R^{(n_0-1, n_0-1)} \right\rangle \left\langle f_0^{(n_0-1, n_0-1)} \right|. \end{aligned}$$

We then define matrices

$$K_i = K_i^{(L)} \otimes K_i^{(R)} \in \text{M}_{n_0} \otimes \text{M}_{k_R+k_L+1}, \quad \tilde{K}_i = \tilde{K}_i^{(L)} \otimes K_i^{(R)} \in \text{M}_{2n_0-1} \otimes \text{M}_{k_R+k_L+1},$$

for  $i = 1, 2, 3$ . We set for each  $t \in [0, 1]$ , matrices  $\tilde{\omega}_\mu(t)$ ,  $\mu = 1, \dots, n$  by

$$\tilde{\omega}_1(t) := R, \quad \tilde{\omega}_2(t) := \left| \chi_1^{(n_0)} \right\rangle \left\langle \eta_0^{(n_0)} \right| + t \left| \eta_0^{(n_0)} \right\rangle \left\langle \chi_1^{(n_0)} \right|, \quad \tilde{\omega}_\mu(t) = 0, \quad \mu \geq 3.$$

We claim that  $\tilde{\omega}(t)$  is primitive, for  $t \in (0, 1]$ . To prove this, we use the same argument as in [BO]. Applying Lemma C.7 of Part I [O1] to the distinct numbers to  $\{r_\alpha\}_{\alpha=2}^{n_0} \cup \{r_\alpha^{-1}\}_{\alpha=2}^{n_0}$ , we obtain  $\varsigma_\alpha = (\varsigma_\alpha(j))_{j=0}^{2n_0-3} \in \mathbb{C}^{2(n_0-1)}$   $\alpha = 2, \dots, n_0$  such that

$$\begin{aligned} \mathcal{K}_l(\tilde{\omega}(t)) &\ni \sum_{j=0}^{2n_0-3} \varsigma_\alpha(j) \tilde{\omega}_1(t)^{l-1-j} \tilde{\omega}_2(t) \tilde{\omega}_1(t)^j = e_{1,\alpha}^{(n_0)}, \\ \mathcal{K}_l(\tilde{\omega}(t)) &\ni \sum_{j=0}^{2n_0-3} \varsigma_\alpha(j) \tilde{\omega}_1(t)^j \tilde{\omega}_2(t) \tilde{\omega}_1(t)^{l-1-j} = t e_{\alpha,1}^{(n_0)}, \end{aligned} \quad (21)$$

for all  $t \in [0, 1]$ ,  $\alpha = 2, \dots, n_0$ , and  $l \geq 2(n_0 - 1)$ . Hence for  $t \in (0, 1]$  and  $l \geq 4(n_0 - 1)$ , we have  $\mathcal{K}_l(\tilde{\omega}(t)) = M_{n_0}$ , i.e.,  $\tilde{\omega}(t)$  is primitive. By Lemma C.6 of Part I [O1], for all  $t \in (0, 1]$ ,  $T_{\tilde{\omega}(t)}$  satisfies all the conditions (1),(2),(3) in Lemma B.3.

For  $t = 0$ , the pentad  $(n, n_0, e_{11}^{(n_0)}, \mathbb{I}, \tilde{\omega}(0))$  satisfies the *Condition 2* (Definition 2.3 [O1]). This can be checked in the same way as the proof of Lemma 3.6 [O1], using (21). Then, by Lemma 2.9 [O1],  $T_{\tilde{\omega}(0)}$  satisfies all the conditions (1), (2), (3) in Lemma B.3. In particular, we have  $r_{T_{\tilde{\omega}(0)}} = 1 > 0$ .

Hence we can apply Lemma B.3 to  $T_{\tilde{\omega}(t)}$  and from the latter Lemma,  $[0, 1] \ni t \mapsto r_{T_{\tilde{\omega}(t)}} \in \mathbb{C}$  is  $C^\infty$ . We define a  $C^\infty$ -path  $[0, 1] \ni t \mapsto \omega(t) \in M_{n_0}^{\times n}$  by

$$\omega_\mu(t) := r_{T_{\tilde{\omega}(t)}}^{-\frac{1}{2}} \tilde{\omega}_\mu(t), \quad t \in [0, 1], \quad \mu = 1, \dots, n.$$

We have  $r_{T_{\omega(t)}} = 1$ , for  $t \in [0, 1]$ . We define a path of  $n$ -tuple of elements in  $M_{n_0} \otimes M_{k_R+k_L+1}$ ,  $\mathbb{B}(t)$  by

$$B_1(t) := \omega_1(t) \otimes \Lambda_{\lambda_R}, \quad B_2(t) := \omega_2(t) \otimes \Lambda_{\lambda_R} + (r_{T_{\omega(t)}})^{-\frac{1}{2}} R \otimes V \Lambda_{\lambda_R}, \quad B_\mu(t) := 0, \quad 3 \leq \mu \leq n, \quad (22)$$

for each  $t \in [0, 1]$ .

**Lemma 4.1.** *Let  $2 \leq n_0 \in \mathbb{N}$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , and  $\mathbb{B}(t)$  be defined by (22). For any  $t \in (0, 1]$ ,  $\mathbb{B}(t)$  belongs to  $\text{Class}(n, n_0, k_R, k_L)$  with  $\lambda_{\mathbb{B}(t)} = \lambda_R$ ,  $\omega_{\mathbb{B}(t)} = \omega(t)$ ,  $D_{a, \mathbb{B}(t)} = E_{-a,0}^{(k_R,0)}$ , and  $G_{b, \mathbb{B}(t)} = E_{0,b}^{(0,k_L)}$ . In particular, by Proposition 3.1 of Part I [O1],  $T_{\mathbb{B}(t)}$  satisfies the Spectral Property II with respect to a triple  $(s_{\mathbb{B}(t)}, e_{\mathbb{B}(t)}, \varphi_{\mathbb{B}(t)})$  and  $0 < s_{\mathbb{B}(t)} < 1$ ,  $s(e_{\mathbb{B}(t)}) = \hat{P}_R^{(n_0, k_R, k_L)}$ ,  $s(\varphi_{\mathbb{B}(t)}) = \hat{P}_L^{(n_0, k_R, k_L)}$ .*

**Proof.** We check that  $\mathbb{B}(t)$  satisfies the condition of  $\text{Class}(n, n_0, k_R, k_L)$  with respect to  $(\lambda_R, \mathbb{D} := (E_{-a,0}^{(k_R,0)})_{a=1}^{k_R}, \mathbb{G} := (E_{0,b}^{(0,k_L)})_{b=1}^{k_L}, 0) \in \mathcal{T}(k_R, k_L)$ . It is trivial that  $(\lambda_R, \mathbb{D} := (E_{-a,0}^{(k_R,0)})_{a=1}^{k_R}, \mathbb{G} := (E_{0,b}^{(0,k_L)})_{b=1}^{k_L}, 0)$  belongs to  $\mathcal{T}(k_R, k_L)$ , with  $\lambda_R \in \text{Wo}'(k_R, k_L)$ . Furthermore, we have  $\mathbb{B}(t) \in (M_{n_0} \otimes \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) \Lambda_{\lambda_R})^{\times n}$ , hence  $\mathcal{K}_l(\mathbb{B}(t)) \subset M_{n_0} \otimes \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) \Lambda_{\lambda_R}^l$  for all  $l \in \mathbb{N}$ . We claim  $\mathcal{K}_l(\mathbb{B}(t)) = M_{n_0} \otimes \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) \Lambda_{\lambda_R}^l$  for  $l$  large enough. As  $\{r_\alpha\}_{\alpha=2}^{n_0} \cup \{\lambda_{Ri}^{-1}\}_{i=-k_R}^{-1} \cup \{r_\alpha^{-1}\}_{\alpha=2}^{n_0} \cup \{\lambda_{Rj}\}_{j=1}^{k_L}$  are distinct, by the routine argument using Lemma C.7 of Part I [O1], we obtain

$$e_{1\alpha}^{(n_0)} \otimes \Lambda_{\lambda_R}^l, e_{\beta,1}^{(n_0)} \otimes \Lambda_{\lambda_R}^l, R^l \otimes E_{-a,0}^{(k_R, k_L)}, R^l \otimes E_{0b}^{(k_R, k_L)} \in \mathcal{K}_l(\mathbb{B}(t)),$$

for  $2 \leq \alpha, \beta \leq n_0$ ,  $a = 1, \dots, k_R$ ,  $b = 1, \dots, k_L$  when  $l$  is large enough. By multiplying these elements, we obtain

$$M_{n_0} \otimes \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) \Lambda_{\lambda_R}^l \subset \mathcal{K}_l(\mathbb{B}(t)),$$

for  $l$  large enough. This proves the claim.  $\square$

Therefore, for  $t \in (0, 1]$ , the Hamiltonians  $H_{m, \Phi(t)}$  are gapped by Theorem 1.18 [O1]. By an analogous argument of Proposition 3.1 [O1], we obtain the following for  $t = 0$ .

**Lemma 4.2.** *Let  $2 \leq n_0 \in \mathbb{N}$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , and  $\mathbb{B}(t)$  be defined by (22). The quadruplet  $(n, n_0(k_R + k_L + 1), e_{11}^{(n_0)} \otimes P_R^{(k_R, k_L)}, \mathbb{I}_{n_0} \otimes P_L^{(k_R, k_L)}, \mathbb{B}(0))$  satisfies the Condition 2 of Definition 2.3 [O1]. In particular, by Lemma 2.9 of Part I [O1],  $T_{\mathbb{B}(0)}$  satisfies the Spectral Property II with respect to a triple  $(s_{\mathbb{B}(0)}, e_{\mathbb{B}(0)}, \varphi_{\mathbb{B}(0)})$  and  $0 < s_{\mathbb{B}(0)} < 1$ ,  $s(e_{\mathbb{B}(0)}) = e_{00}^{(n_0)} \otimes P_R^{(k_R, k_L)}$ ,  $s(\varphi_{\mathbb{B}(0)}) = \hat{P}_L^{(n_0, k_R, k_L)}$ .*

From the last two Lemmas, we can apply Lemma B.3 and obtain the following.

**Lemma 4.3.** *There exists an  $0 < s_1 < 1$  such that  $\sigma(T_{\mathbb{B}(t)}) \setminus \{1\} \subset \mathcal{B}_{s_1}(0)$ ,  $t \in [0, 1]$ .*

## 4.2 The $t \downarrow 0$ limit

Throughout this subsection, let  $2 \leq n_0 \in \mathbb{N}$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , and  $\mathbb{B}(t)$  be defined by (22). The path  $\mathbb{B}(t)$  has a singularity, i.e., although  $\mathbb{B}(t)$  belongs to  $\text{Class}(n, n_0, k_R, k_L)$  for  $t \in (0, 1]$ , its  $t \downarrow 0$  limit  $\mathbb{B}(0)$  does not. In this subsection, we consider the  $t \downarrow 0$  limit of  $\mathcal{G}_{l, \mathbb{B}(t)}$ . To describe the limit, we need some preparation. We define  $n$ -tuple  $\mathbb{A}$  of elements in  $M_{2n_0-1} \otimes M_{k_R+k_L+1}$  by

$$A_1 := \Lambda_{\lambda_L} \otimes \Lambda_{\lambda_R}, \quad A_2 := \tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3, \quad A_\mu := 0, \quad 3 \leq \mu \leq n. \quad (23)$$

We set projections

$$P_{R, \mathbb{A}} := P_R^{(n_0-1, n_0-1)} \otimes P_R^{(k_R, k_L)}, \quad P_{L, \mathbb{A}} := P_L^{(n_0-1, n_0-1)} \otimes P_L^{(k_R, k_L)}. \quad (24)$$

Furthermore, for  $\alpha = -(n_0 - 1), \dots, n_0 - 1$  and  $i = -k_R, \dots, k_L$ , we denote

$$g_{\alpha, i, \mathbb{A}} := f_\alpha^{(n_0-1, n_0-1)} \otimes f_i^{(k_R, k_L)}. \quad (25)$$

For each  $l, k \in \mathbb{N}$  with  $k \leq l$  and  $i_1, i_2, \dots, i_k \in \{1, 2, 3\}$ , define  $\zeta_{i_1 \dots i_k}^{(k)l} : M_{n_0} \otimes M_{k_R+k_L+1} \rightarrow \bigotimes_{i=0}^{l-1} \mathbb{C}^n$  by

$$\zeta_{i_1 \dots i_k}^{(k)l}(X) := \sum_{1 \leq m_1 < \dots < m_k \leq l} \text{Tr} \left( X (B_1(0))^{m_1-1} K_{i_1} B_1(0)^{m_2-m_1-1} K_{i_2} \dots B_1(0)^{m_k-m_{k-1}-1} K_{i_k} B_1(0)^{l-m_k} \right)^* \\ \psi_1^{\otimes m_1-1} \otimes \psi_2 \otimes \psi_1^{\otimes m_2-m_1-1} \otimes \psi_2 \otimes \dots \otimes \psi_1^{\otimes m_k-m_{k-1}-1} \otimes \psi_2 \otimes \psi_1^{\otimes l-m_k}, \quad (26)$$

for  $X \in M_{n_0} \otimes M_{k_R+k_L+1}$ . For each  $4 \leq l \in \mathbb{N}$ , define  $\Upsilon_l : M_{k_R+k_L+1} \rightarrow \bigotimes_{i=0}^{l-1} \mathbb{C}^n$  by

$$\Upsilon_l(x) := \left( \zeta_{13}^{(2)l} + \zeta_{213}^{(3)l} + \zeta_{123}^{(3)l} + \zeta_{132}^{(3)l} + \zeta_{2213}^{(4)l} + \zeta_{2123}^{(4)l} + \zeta_{2132}^{(4)l} + \zeta_{1223}^{(4)l} + \zeta_{1232}^{(4)l} + \zeta_{1322}^{(4)l} \right) \left( e_{11}^{(n_0)} \otimes x \right), \\ x \in M_{k_R+k_L+1}. \quad (27)$$

For  $l \in \mathbb{N}$ ,  $\alpha, \beta \in \{1, \dots, n_0\}$ ,  $a = 0, 1, \dots, k_R$ , and  $b = 0, 1, \dots, k_L$ , we define

$$\xi_{\alpha, \beta, a, b}^{(l)} := \Gamma_{l, \mathbb{A}}^{(R)} \left( E_{-(\alpha-1), \beta-1}^{(n_0-1, n_0-1)} \otimes E_{-a, b}^{(k_R, k_L)} \right) - \delta_{\alpha, \beta} (1 - \delta_{\alpha, 1}) \bar{r}_\alpha^l \Upsilon_l(E_{-a, b}^{(k_R, k_L)}). \quad (28)$$

For each  $l \in \mathbb{N}$ , we denote the linear subspace of  $\bigotimes_{i=0}^{l-1} \mathbb{C}^n$  spanned by  $\xi_{\alpha, \beta, a, b}^{(l)}$ ,  $\alpha, \beta \in \{1, \dots, n_0\}$ ,  $a = 0, 1, \dots, k_R$ , and  $b = 0, 1, \dots, k_L$ , by  $\mathcal{D}_l$ . Furthermore, we denote the orthogonal projection onto  $\mathcal{D}_l$  by  $G_l$ , i.e.,

$$G_l := \text{orthogonal projection onto } \text{span}\{\xi_{\alpha, \beta, a, b}^{(l)}\}_{\alpha, \beta=1, \dots, n_0, a=0, 1, \dots, k_R, b=0, 1, \dots, k_L}. \quad (29)$$

Set  $\widehat{e_{11}^{(n_0)}} := e_{11}^{(n_0)} \otimes \mathbb{I}_{k_R+k_L+1}$  and  $a_t := \widehat{e_{11}^{(n_0)}} + \frac{1}{t} \overline{\widehat{e_{11}^{(n_0)}}}$ , for each  $t \in (0, 1]$ . For each  $l \in \mathbb{N}$ , and  $t \in (0, 1]$ , we define maps  $\varpi_l, \Xi_l, \Theta_{l,t} : M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)} \rightarrow M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ , by

$$\varpi_l(X) := e_{11}^{(n_0)} \otimes \text{Tr}_{n_0} \left( \left( \overline{e_{11}^{(n_0)}} R^{*l} \otimes \mathbb{I}_{k_R+k_L+1} \right) X \right), \quad \Xi_l(X) := \overline{\widehat{e_{11}^{(n_0)}}} X - \varpi_l(X), \quad \Theta_{l,t}(X) := \widehat{e_{11}^{(n_0)}} X + \frac{1}{t} \Xi_l(X), \quad (30)$$

for  $X \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ . Here,  $\text{Tr}_{n_0} : M_{n_0} \otimes M_{k_R+k_L+1} \rightarrow M_{k_R+k_L+1}$  indicates the partial trace. Note that  $\Theta_{l,t}$  is a bijection from  $M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$  onto itself. We have

$$\lim_{t \rightarrow 0} \Theta_{l,t}(X) = a_t X, \quad X \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}, \quad t \in (0, 1]. \quad (31)$$

Now we consider the  $t \downarrow 0$  limit.

**Lemma 4.4.** *For each  $l \in \mathbb{N}$ ,  $\alpha, \beta \in \{1, \dots, n_0\}$ ,  $a = 0, 1, \dots, k_R$ , and  $b = 0, 1, \dots, k_L$ , we have the limit*

$$\lim_{t \downarrow 0} \Gamma_{l, \mathbb{B}(t)}^{(R)} \left( \Theta_{l,t} \left( e_{\alpha\beta}^{(n_0)} \otimes E_{-a,b}^{(k_R, k_L)} \right) \right) = \xi_{\alpha, \beta, a, b}^{(l)}. \quad (32)$$

**Proof.** First, we have  $\lim_{t \downarrow 0} \Gamma_{l, \mathbb{B}(t)}^{(R)} \left( e_{1\beta}^{(n_0)} \otimes E_{-a,b}^{(k_R, k_L)} \right) = \Gamma_{l, (B_1(0), B_2(0), 0, \dots, 0)}^{(R)} \left( e_{1\beta}^{(n_0)} \otimes E_{-a,b}^{(k_R, k_L)} \right)$ . Let us consider the monomials  $B_{\mu^{(l)}}(0)$ ,  $\mu^{(l)} \in \{1, \dots, n\}^{\times l}$  which appears in the representation of  $\Gamma_{l, \mathbb{B}(0)}^{(R)}$ . Note that  $B_1(0) = R \otimes \Lambda_{\lambda_R}$  is diagonal and  $B_2(0) = K_1 + K_2$ . Therefore, monomials of  $B_1(0)$  and  $B_2(0)$  are sum of monomials of  $B_1(0)$ ,  $K_1$ , and  $K_2$ . A monomial of  $B_1(0)$ ,  $K_1$ , and  $K_2$  vanishes if there are more than one  $K_1$ . (Note  $K_1^{(L)} R^{l-1} K_1^{(L)} = 0$ ,  $l \in \mathbb{N}$ .) It also vanishes if there are more than two  $K_2$ . (Note  $V \Lambda_{\lambda_R}^{l_1} V \Lambda_{\lambda_R}^{l_2} V = 0$ ,  $l_1, l_2 \in \mathbb{N}$ .) Furthermore, if  $\beta \neq 1$ , in order to give a nontrivial contribution, the monomial has to own one  $K_1$ . On the other hand, if  $\beta = 1$ , the number of  $K_1$  should be zero. From this kind of observations we obtain

$$\begin{aligned} & \lim_{t \downarrow 0} \Gamma_{l, \mathbb{B}(t)}^{(R)} \left( e_{1\beta}^{(n_0)} \otimes E_{-a,b}^{(k_R, k_L)} \right) \\ &= \begin{cases} \left( \text{Tr } E_{-a,b}^{(k_R, k_L)} (\Lambda_{\lambda_R}^*)^l \right) \psi_1^{\otimes l} + \left( \zeta_2^{(1)(l)} + \zeta_{2,2}^{(2)(l)} \right) \left( e_{11}^{(n_0)} \otimes E_{-a,b}^{(k_R, k_L)} \right), & \text{if } \beta = 1, \\ \left( \zeta_1^{(1)(l)} + \zeta_{21}^{(2)(l)} + \zeta_{12}^{(2)(l)} + \zeta_{221}^{(3)(l)} + \zeta_{212}^{(3)(l)} + \zeta_{122}^{(3)(l)} \right) \left( e_{1\beta}^{(n_0)} \otimes E_{-a,b}^{(k_R, k_L)} \right), & \text{if } \beta \in \{2, \dots, n_0\}. \end{cases} \end{aligned}$$

Similar kind of observation about  $\mathbb{A}$  shows that the right hand side coincides with  $\Gamma_{l, \mathbb{A}}^{(R)} \left( E_{0, \beta-1}^{(n_0-1, n_0-1)} \otimes E_{-a,b}^{(k_R, k_L)} \right)$ .

Next let us consider the case with  $\alpha \neq 1$  and  $\alpha \neq \beta$ . Note that in the representation of  $\frac{1}{t} \Gamma_{l, \mathbb{B}(t)}^{(R)} \left( e_{\alpha\beta}^{(n_0)} \otimes E_{-a,b}^{(k_R, k_L)} \right)$ , terms with more than two  $r_{T_{\omega(t)}}^{-\frac{1}{2}} t K_3$  vanish in the  $t \downarrow 0$  limit. On the other hand, all the monomials of  $B_1(0)$ ,  $K_1$ , and  $K_2$  are either of the form "diagonal matrix  $\otimes M_{k_R+k_L+1}$ " or in  $(e_{11}^{(n_0)} M_{n_0}) \otimes M_{k_R+k_L+1}$ . Therefore, we have

$$\Gamma_{l, (B_1(0), B_2(0), 0, \dots)}^{(R)} \left( e_{\alpha, \beta}^{(n_0)} \otimes E_{-a,b}^{(k_R, k_L)} \right) = 0,$$

because  $\alpha \neq \beta$  and  $\alpha \neq 1$ . Hence, what is left is only the monomials of  $B_1(0)$ ,  $K_1$ ,  $K_2$  and  $K_3$  with exactly one  $K_3$ . The contribution of such a monomial is zero if there exists  $K_1$  left to  $K_3$ . This is because such a term belongs to  $(e_{11}^{(n_0)} M_{n_0}) \otimes M_{k_R+k_L+1}$ . Terms with two  $K_1$  without having  $K_3$  between vanish by the same argument as  $\alpha = 1$  case. Also, terms with more than two  $K_2$  also

disappear. From these observation we obtain for  $\alpha \neq \beta$ ,

$$\begin{aligned} & \lim_{t \downarrow 0} \frac{1}{t} \Gamma_{l, \mathbb{B}(t)}^{(R)} \left( e_{\alpha\beta}^{(n_0)} \otimes E_{-a,b}^{(k_R, k_L)} \right) \\ &= \left( \zeta_3^{(1)(l)} + \zeta_{23}^{(2)(l)} + \zeta_{32}^{(2)(l)} + \zeta_{223}^{(3)(l)} + \zeta_{232}^{(3)(l)} + \zeta_{322}^{(3)(l)} + \zeta_{31}^{(2)(l)} + \zeta_{231}^{(3)(l)} + \zeta_{321}^{(3)(l)} + \zeta_{312}^{(3)(l)} \right) \left( e_{\alpha\beta}^{(n_0)} \otimes E_{-a,b}^{(k_R, k_L)} \right) \\ &+ \left( \zeta_{2231}^{(4)(l)} + \zeta_{2321}^{(4)(l)} + \zeta_{2312}^{(4)(l)} + \zeta_{3221}^{(4)(l)} + \zeta_{3212}^{(4)(l)} + \zeta_{3122}^{(4)(l)} \right) \left( e_{\alpha\beta}^{(n_0)} \otimes E_{-a,b}^{(k_R, k_L)} \right). \end{aligned}$$

Similar observation about  $\mathbb{A}$  shows that the right hand side coincides with  $\Gamma_{l, \mathbb{A}}^{(R)} \left( E_{-(\alpha-1), \beta-1}^{(n_0-1, n_0-1)} \otimes E_{-a,b}^{(k_R, k_L)} \right)$ .

Finally, we consider the case with  $\alpha \neq 1$  and  $\alpha = \beta$ . Note that terms in  $\frac{1}{t} \Gamma_{l, \mathbb{B}(t)}^{(R)} \left( \left( e_{\alpha\alpha}^{(n_0)} - \bar{r}_\alpha^l e_{11}^{(n_0)} \right) \otimes E_{-a,b}^{(k_R, k_L)} \right)$  with more than two  $r_{T_{\omega(t)}}^{-\frac{1}{2}} t K_3$  vanish in the  $t \downarrow 0$  limit. Now let us see that the terms given by monomials of  $B_1(0)$ ,  $K_1$ , and  $K_2$  vanish. There can be at most one  $K_1$  for nonzero monomial of  $B_1(0)$ ,  $K_1$ , and  $K_2$  as above. However, if there is one, then the monomial belongs to  $\left( e_{11}^{(n_0)} \overline{M_{n_0} e_{11}^{(n_0)}} \right) \otimes M_{k_R+k_L+1}$ . Therefore, these terms are orthogonal to  $\left( e_{\alpha, \alpha}^{(n_0)} - \bar{r}_\alpha^l e_{11}^{(n_0)} \right) \otimes M_{k_R+k_L+1}$  with respect to the inner product  $\langle \cdot, \cdot \rangle_{\text{Tr}}$  given by  $\langle X, Y \rangle_{\text{Tr}} = \text{Tr } X^* Y$ . Hence we conclude that the number of  $K_1$  should be zero. In case there is no  $K_1$ , the monomial belongs to  $R^l \otimes M_{k_R+k_L+1}$ . But such an element is again orthogonal to  $\left( e_{\alpha, \alpha}^{(n_0)} - \bar{r}_\alpha^l e_{11}^{(n_0)} \right) \otimes M_{k_R+k_L+1}$  with respect to  $\langle \cdot, \cdot \rangle_{\text{Tr}}$ . Hence any term given by a monomial of  $B_1(0)$ ,  $K_1$ , and  $K_2$  vanishes.

For the reasons above, terms which are left in  $t \downarrow 0$  limit are only the monomials of  $B_1(0)$ ,  $K_1$ ,  $K_2$  and  $K_3$  with exactly one  $K_3$ . Terms with two  $K_1$  without having  $K_3$  in between vanish by observation in  $\alpha \neq 1$ ,  $\alpha \neq \beta$  case. Therefore, there can be at most two  $K_1$ . Under these conditions, if furthermore there is no  $K_1$  in the monomial, it belongs to  $\overline{e_{11}^{(n_0)} M_{n_0} e_{11}^{(n_0)}} \otimes M_{k_R+k_L+1}$ . If there are two  $K_1$  in the monomial, then from the above observation, they have to be ordered as  $K_1, K_3, K_1$ , and the monomial belongs to  $e_{11}^{(n_0)} \overline{M_{n_0} e_{11}^{(n_0)}} \otimes M_{k_R+k_L+1}$ . In both cases, the monomial is orthogonal to  $\left( e_{\alpha, \alpha}^{(n_0)} - \bar{r}_\alpha^l e_{11}^{(n_0)} \right) \otimes M_{k_R+k_L+1}$  with respect to  $\langle \cdot, \cdot \rangle_{\text{Tr}}$ . Therefore, to have a non-zero contribution, the number of  $K_1$  has to be one. When there is one  $K_1$ , the monomial belongs to  $\left( e_{11}^{(n_0)} \overline{M_{n_0} e_{11}^{(n_0)}} \right) \otimes M_{k_R+k_L+1}$  if  $K_1$  is left to  $K_3$ , and it belongs to  $\left( \overline{e_{11}^{(n_0)} M_{n_0} e_{11}^{(n_0)}} \right) \otimes M_{k_R+k_L+1}$  if  $K_3$  is left to  $K_1$ . Terms with more than two  $K_2$ s disappear as usual. From these observations, we obtain for  $\alpha \neq \beta$ ,

$$\begin{aligned} & \lim_{t \downarrow 0} \frac{1}{t} \Gamma_{l, \mathbb{B}(t)}^{(R)} \left( \left( e_{\alpha\alpha}^{(n_0)} - \bar{r}_\alpha^l e_{11}^{(n_0)} \right) \otimes E_{-a,b}^{(k_R, k_L)} \right) \\ &= \left( \zeta_{31}^{(2)(l)} + \zeta_{231}^{(3)(l)} + \zeta_{321}^{(3)(l)} + \zeta_{312}^{(3)(l)} \right) \left( e_{\alpha\alpha}^{(n_0)} \otimes E_{-a,b}^{(k_R, k_L)} \right) \\ &+ \left( \zeta_{2231}^{(4)(l)} + \zeta_{2321}^{(4)(l)} + \zeta_{2312}^{(4)(l)} + \zeta_{3221}^{(4)(l)} + \zeta_{3212}^{(4)(l)} + \zeta_{3122}^{(4)(l)} \right) \left( e_{\alpha\alpha}^{(n_0)} \otimes E_{-a,b}^{(k_R, k_L)} \right) \\ &- \bar{r}_\alpha^l \left( \zeta_{13}^{(2)(l)} + \zeta_{213}^{(3)(l)} + \zeta_{123}^{(3)(l)} + \zeta_{132}^{(3)(l)} \right) \left( e_{11}^{(n_0)} \otimes E_{-a,b}^{(k_R, k_L)} \right) \\ &- \bar{r}_\alpha^l \left( \zeta_{2213}^{(4)(l)} + \zeta_{2123}^{(4)(l)} + \zeta_{2132}^{(4)(l)} + \zeta_{1223}^{(4)(l)} + \zeta_{1232}^{(4)(l)} + \zeta_{1322}^{(4)(l)} \right) \left( e_{11}^{(n_0)} \otimes E_{-a,b}^{(k_R, k_L)} \right). \end{aligned}$$

It is straight forward to check that the right hand side coincides with  $\Gamma_{l, \mathbb{A}}^{(R)} \left( E_{-(\alpha-1), \alpha-1}^{(n_0-1, n_0-1)} \otimes E_{-a,b}^{(k_R, k_L)} \right) - \delta_{\alpha, \alpha} (1 - \delta_{\alpha, 1}) \bar{r}_\alpha^l \Upsilon_l(E_{-a,b}^{(k_R, k_L)})$ .  $\square$

### 4.3 The properties of $\mathbb{A}$

Throughout this subsection, let  $2 \leq n_0 \in \mathbb{N}$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , and  $\mathbb{A}$  be given by (23). By a routine argument, using Lemma C.7 of [O1], we obtain the following Lemma.

**Lemma 4.5.** *Set*

$$\mathcal{D}_{\mathbb{A}} := \text{span} \left( \left\{ \mathbb{I}, E_{-\alpha,0}^{(n_0-1,n_0-1)}, E_{0,\beta}^{(n_0-1,n_0-1)}, E_{-\alpha,\beta}^{(k_R,k_L)} \right\}_{\alpha,\beta=1,\dots,n_0-1} \otimes \left\{ \mathbb{I}, E_{-a,0}^{(k_R,k_L)}, E_{0,b}^{(k_R,k_L)}, E_{-a,b}^{(k_R,k_L)} \right\}_{a=1,\dots,k_R,b=1,\dots,k_L} \right).$$

There exists an  $l_{\mathbb{A}} \in \mathbb{N}$  such that

$$\mathcal{K}_l(\mathbb{A}) = \mathcal{D}_{\mathbb{A}} (\Lambda_{\lambda_L}^l \otimes \Lambda_{\lambda_R}^l),$$

for all  $l \geq l_{\mathbb{A}}$ .

Using this, we obtain the following Lemma. The proof is the same as that of Proposition 3.1 [O1].

**Lemma 4.6.** *The pentad  $(n, (2n_0-1)(k_L+k_R+1), P_{R,\mathbb{A}}, P_{L,\mathbb{A}}, \mathbb{A})$  satisfies Condition 2, defined in Definition 2.3 [O1]. Therefore, by Lemma 2.9 [O1], there exist a number  $0 < s_{\mathbb{A}} < 1$ , a state  $\varphi_{\mathbb{A}}$  on  $M_{2n_0-1} \otimes M_{k_R+k_L+1}$  and a positive element  $e_{\mathbb{A}} \in (M_{2n_0-1} \otimes M_{k_R+k_L+1})_+$ , such that  $T_{\mathbb{A}}$  satisfies the Spectral Property II with respect to  $(s_{\mathbb{A}}, e_{\mathbb{A}}, \varphi_{\mathbb{A}})$  (see Definition 2.7 [O1]) and  $s(e_{\mathbb{A}}) = P_{R,\mathbb{A}}$ , and  $s(\varphi_{\mathbb{A}}) = P_{L,\mathbb{A}}$ . Furthermore, the pentad  $(n, (2n_0-1)(k_L+k_R+1), P_{R,\mathbb{A}}, P_{L,\mathbb{A}}, \mathbb{A})$  satisfies Condition 3 for  $l_{\mathbb{A}}$ , and the triple  $(n, (2n_0-1)(k_L+k_R+1), \mathbb{A})$  satisfies Condition 4, for  $(l_{\mathbb{A}}, l_{\mathbb{A}})$ . Therefore, by Lemma 2.14, Proposition 2.6 of [O1],  $m_{\mathbb{A}} \leq 2l_{\mathbb{A}} < \infty$  and  $M_{\mathbb{A},P_{R,\mathbb{A}},P_{L,\mathbb{A}}} < \infty$ . (Here  $m_{\mathbb{A}}$ ,  $M_{\mathbb{A},P_{R,\mathbb{A}},P_{L,\mathbb{A}}}$  defined in Definition 1.4 and (8) of [O1].) From Proposition 2.6 [O1],  $\mathcal{G}_{l,\mathbb{A}}$  satisfies Condition 1 (see Definition 2.1 of [O1]).*

By the argument parallel to Lemma 3.11-13 and Lemma 3.15, 3.16 of [O1], we obtain the following.

**Lemma 4.7.** *There exists a completely positive map from the half-infinite chain  $\mathcal{A}_{(-\infty,-1]}$  to  $M_{2n_0-1} \otimes M_{k_R+k_L+1}$ , such that*

$$\mathbb{L}_{\mathbb{A}}(B) := \sum_{\mu^{(l)}, \nu^{(l)} \in \{1,\dots,n\} \times l} \left\langle \widehat{\psi_{\mu^{(l)}}}, B \widehat{\psi_{\nu^{(l)}}} \right\rangle \left( \widehat{A_{\nu^{(l)}}} \right)^* \rho_{\mathbb{A}} \widehat{A_{\mu^{(l)}}},$$

if  $B \in \mathcal{A}_{[-l,-1]} \simeq \otimes_{i=0}^{l-1} M_n$  for  $l \in \mathbb{N}$ . We have

$$\text{Ran } \mathbb{L}_{\mathbb{A}} = P_{L,\mathbb{A}} (M_{2n_0-1} \otimes M_{k_R+k_L+1}) P_{L,\mathbb{A}}. \quad (33)$$

For each  $\sigma_L \in \mathfrak{E}_{n_0(k_L+1)}$ , under the identification  $M_{n_0(k_L+1)} \simeq P_{L,\mathbb{A}} (M_{2n_0-1} \otimes M_{k_R+k_L+1}) P_{L,\mathbb{A}}$ , define  $\Xi_{L,\mathbb{A}}(\sigma_L) : \mathcal{A}_{(-\infty,-1]} \rightarrow \mathbb{C}$  by

$$\Xi_{L,\mathbb{A}}(\sigma_L)(B) := \sigma_L(y_{\mathbb{A}}^{\frac{1}{2}} \mathbb{L}_{\mathbb{A}}(B) y_{\mathbb{A}}^{\frac{1}{2}}), \quad B \in \mathcal{A}_{(-\infty,-1]}.$$

Then for  $m \geq m_{\mathbb{A}}$ , the map  $\Xi_{L,\mathbb{A}} : \mathfrak{E}_{n_0(k_L+1)} \rightarrow \mathcal{S}_{(-\infty,-1]}(H_{\Phi_{m,\mathbb{A}}})$  defines an affine bijection. An analogous statement holds for the right half infinite chain and defines  $\Xi_{R,\mathbb{A}}, \mathbb{R}_{\mathbb{A}}$ . The set  $\mathcal{S}_{\mathbb{Z}}(H_{\Phi_{m,\mathbb{A}}})$  consists of a unique state

$$\omega_{\mathbb{A},\infty} = \bigotimes_{\mathbb{Z}} \rho_0, \quad (34)$$

where  $\rho_0$  is a state on  $M_n$  such that  $\rho_0 = \left\langle \chi_1^{(n)}, (\cdot) \chi_1^{(n)} \right\rangle$ .

The statement of Lemma 3.25 of [O1] also holds for  $\mathbb{A}$ .

**Lemma 4.8.** *Let  $m \geq m_{\mathbb{A}}$ . There exists a constant  $C_{\mathbb{A}}''' > 0$  satisfying the following. : Let  $M \in \mathbb{N}$  and  $\varphi$  be a state on  $\mathcal{A}_{\mathbb{Z}}$ . Assume that we have  $\varphi(\tau_i(1 - G_{m,\mathbb{A}})) = 0$  for all  $i \in \mathbb{Z}$  with  $[i, i + m - 1] \subset [-M, M]^c$ . Then for any  $L \in \mathbb{N}$  with  $M + 1 \leq L$  and  $A \in \mathcal{A}_{[-L+1, L-1]^c}$ , we have*

$$\left| \varphi(A) - \left( \omega_{\mathbb{A},\infty}|_{\mathcal{A}_{(-\infty,-1]}} \otimes \omega_{\mathbb{A},\infty}|_{\mathcal{A}_{[0,\infty)}} \right)(A) \right| \leq C_{\mathbb{A}}''' s_{\mathbb{A}}^{L-M} \|A\|. \quad (35)$$

Furthermore, as in Lemma 3.15-22 of [O1], we obtain the followings.

**Lemma 4.9.** *[A1]-[A5] in [O2] hold for  $H_{\Phi_{m,\mathbb{A}}}$  for  $m \geq 2l_{\mathbb{A}}$ .*

**Proof.** The proof is a straight forward application of the argument of Part I [O1].  $\square$

**Notation 4.10.** From Theorem 1.2 of [O2], for each  $m \geq 2l_{\mathbb{A}}$ , there exists a  $\mathbb{V} \in \text{ClassA}$  with respect to some  $(n_{0,\mathbb{V}}, k_{R,\mathbb{V}}, k_{L,\mathbb{V}}, \lambda_{\mathbb{V}}, \mathbb{D}_{\mathbb{V}}, \mathbb{G}_{\mathbb{V}}, Y_{\mathbb{V}})$  such that

$$\mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m,\mathbb{A}}}) = \mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m_1,\mathbb{V}}}), \quad \mathcal{S}_{[0,\infty)}(H_{\Phi_{m,\mathbb{A}}}) = \mathcal{S}_{[0,\infty)}(H_{\Phi_{m_1,\mathbb{V}}}), \quad \mathcal{S}_{\mathbb{Z}}(H_{\Phi_{m,\mathbb{A}}}) = \mathcal{S}_{\mathbb{Z}}(H_{\Phi_{m_1,\mathbb{V}}}) = \{\omega_{\mathbb{V},\infty}\} = \{\omega_{\mathbb{A},\infty}\}, \quad (36)$$

for all  $m_1 \geq 2l_{\mathbb{V}}$ . Furthermore,  $H_{\Phi_{m,\mathbb{A}}}$  and  $H_{\Phi_{m_1,\mathbb{V}}}$  are type II- $C^1$ -equivalent, by Corollary 1.4 of [O2].

**Lemma 4.11.** *This  $\mathbb{V}$  in Notation 4.10 belongs to  $\text{Class}(n, 1, n_0(k_R + 1) - 1, n_0(k_L + 1) - 1)$ .*

**Proof.** We already know that  $\mathbb{V} \in \text{ClassA}$ . Therefore, there exist  $n_{0,\mathbb{V}} \in \mathbb{N}$ ,  $k_{R\mathbb{V}}, k_{L\mathbb{V}} \in \mathbb{N} \cup \{0\}$ ,  $(\lambda_{\mathbb{V}}, \mathbb{D}_{\mathbb{V}}, \mathbb{G}_{\mathbb{V}}, Y_{\mathbb{V}}) \in \mathcal{T}(k_{R\mathbb{V}}, k_{L\mathbb{V}})$  such that  $\mathbb{V} \in \mathfrak{B}(n, n_{0,\mathbb{V}}, k_{R\mathbb{V}}, k_{L\mathbb{V}}, \lambda_{\mathbb{V}}, \mathbb{D}_{\mathbb{V}}, \mathbb{G}_{\mathbb{V}}, Y_{\mathbb{V}})$ . We have to show that  $n_{0,\mathbb{V}} = 1$ ,  $k_{R\mathbb{V}} = n_0(k_R + 1) - 1$ ,  $k_{L\mathbb{V}} = n_0(k_L + 1) - 1$ ,  $Y_{\mathbb{V}} = 0$ , and  $\lambda_{\mathbb{V}} \in \text{Wo}'(k_{R\mathbb{V}}, k_{L\mathbb{V}})$ .

First we show  $n_{0,\mathbb{V}} = 1$ . Recall from Part I [O1] Lemma 3.2, we obtain  $\omega_{\mathbb{V}} \in \text{Prim}_1(n, n_{0,\mathbb{V}})$  by

$$\omega_{\mu,\mathbb{V}} \otimes E_{00}^{(k_R, k_L)} = \left( \mathbb{I} \otimes E_{00}^{(k_R, k_L)} \right) V_{\mu} \left( \mathbb{I} \otimes E_{00}^{(k_R, k_L)} \right), \quad \mu = 1, \dots, n.$$

By the primitivity, from C.6 of [O1], there exist a faithful positive linear functional  $\varphi_{\omega_{\mathbb{V}}}$  on  $M_{n_{0,\mathbb{V}}}$  and a strictly positive element  $e_{\omega_{\mathbb{V}}} \in M_{n_{0,\mathbb{V}}}$  such that  $\lim_{N \rightarrow \infty} T_{\omega_{\mathbb{V}}}^N(A) = \varphi_{\omega_{\mathbb{V}}}(A)e_{\omega_{\mathbb{V}}}$  for all  $A \in M_{n_{0,\mathbb{V}}}$ . In particular, we have  $\varphi_{\omega_{\mathbb{V}}}(e_{\omega_{\mathbb{V}}}) = 1$ .

On the other hand, by Lemma 2.9 [O1], we have  $\lim_{N \rightarrow \infty} T_{\mathbb{V}}^N(A) = \varphi_{\mathbb{V}}(A)e_{\mathbb{V}}$  for all  $A \in M_{n_{0,\mathbb{V}}} \otimes M_{k_{R\mathbb{V}} + k_{L\mathbb{V}} + 1}$ . Sandwiching this by  $\widehat{E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}} = \mathbb{I} \otimes E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}$ , we obtain

$$\begin{aligned} & \varphi_{\mathbb{V}} \left( X \otimes E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})} \right) \cdot \widehat{E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}} e_{\mathbb{V}} \widehat{E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}} \\ &= \lim_{N \rightarrow \infty} \widehat{E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}} T_{\mathbb{V}}^N \left( X \otimes E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})} \right) \widehat{E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}} = \lim_{N \rightarrow \infty} T_{\omega_{\mathbb{V}}}^N(X) \otimes E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})} \\ &= \varphi_{\omega_{\mathbb{V}}}(X) e_{\omega_{\mathbb{V}}} \otimes E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}, \end{aligned} \quad (37)$$

for any  $X \in M_{n_{0,\mathbb{V}}}$ . In the second equality, we used the property  $\widehat{P_L^{(n_{0,\mathbb{V}}, k_{R\mathbb{V}}, k_{L\mathbb{V}})}} \widehat{V_{\mu^{(l)}}} \widehat{P_R^{(n_{0,\mathbb{V}}, k_{R\mathbb{V}}, k_{L\mathbb{V}})}} = \widehat{\omega_{\mu^{(l)}, \mathbb{V}}} \otimes E_{00}^{(n_{0,\mathbb{V}}, k_{R\mathbb{V}}, k_{L\mathbb{V}})}$ , for  $\mu^{(l)} \in \{1, \dots, n\}^{\times l}$ . Substituting  $X = e_{\omega_{\mathbb{V}}}$  to this equality, we obtain

$$e_{\omega_{\mathbb{V}}} \otimes E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})} = \varphi_{\mathbb{V}} \left( e_{\omega_{\mathbb{V}}} \otimes E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})} \right) \cdot \widehat{E_{00}^{(n_{0,\mathbb{V}}, k_{R\mathbb{V}}, k_{L\mathbb{V}})}} e_{\mathbb{V}} \widehat{E_{00}^{(n_{0,\mathbb{V}}, k_{R\mathbb{V}}, k_{L\mathbb{V}})}}. \quad (38)$$



We used  $\varphi_{\omega_{\mathbb{V}}}(e_{\omega_{\mathbb{V}}}) = 1$  on the left hand side. Substituting (38) to (37), we obtain

$$\varphi_{\omega_{\mathbb{V}}}(X) = \frac{\varphi_{\mathbb{V}}(X \otimes E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})})}{\varphi_{\mathbb{V}}(e_{\omega_{\mathbb{V}}} \otimes E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})})}, \quad X \in M_{n_0, \mathbb{V}}. \quad (39)$$

We set  $\mathbf{u} \in M_{n_0, \mathbb{V}}^{\times n}$  by  $u_{\mu} := e_{\omega_{\mathbb{V}}}^{-\frac{1}{2}}(\omega_{\mu, \mathbb{V}}) e_{\omega_{\mathbb{V}}}^{\frac{1}{2}}, \mu = 1, \dots, n$ . Then  $T_{\mathbf{u}}$  is a unital CP map on  $M_{n_0, \mathbb{V}}$  and  $\varphi_{\mathbf{u}} := \varphi_{\omega_{\mathbb{V}}} \circ \text{Ad } e_{\omega_{\mathbb{V}}}^{\frac{1}{2}}$  is a faithful  $T_{\mathbf{u}}$ -invariant state on  $M_{n_0, \mathbb{V}}$ . From Lemma 3.16 of [O1] and (38), (39), the unique element  $\omega_{\mathbb{V}, \infty}$  in  $\mathcal{S}_{\mathbb{Z}}(H_{\Phi_{m, \mathbb{A}}}) = \mathcal{S}_{\mathbb{Z}}(H_{\Phi_{m_1, \mathbb{V}}})$  is

$$\begin{aligned} \omega_{\mathbb{V}, \infty}(A) &= \sum_{\mu^{(l)}, \nu^{(l)} \in \{1, \dots, n\} \times l} \langle \widehat{\psi_{\mu^{(l)}}}, A \widehat{\psi_{\nu^{(l)}}} \rangle \varphi_{\mathbb{V}}(\widehat{V_{\mu^{(l)}}} e_{\mathbb{V}}(\widehat{V_{\nu^{(l)}}})^*) \\ &= \sum_{\mu^{(l)}, \nu^{(l)} \in \{1, \dots, n\} \times l} \langle \widehat{\psi_{\mu^{(l)}}}, A \widehat{\psi_{\nu^{(l)}}} \rangle \varphi_{\mathbb{V}}\left(P_L^{(n_0, \mathbb{V}, k_{R\mathbb{V}}, k_{L\mathbb{V}})} \widehat{V_{\mu^{(l)}}} P_R^{(n_0, \mathbb{V}, k_{R\mathbb{V}}, k_{L\mathbb{V}})} e_{\mathbb{V}} P_R^{(n_0, \mathbb{V}, k_{R\mathbb{V}}, k_{L\mathbb{V}})} (\widehat{V_{\nu^{(l)}}})^* P_L^{(n_0, \mathbb{V}, k_{R\mathbb{V}}, k_{L\mathbb{V}})}\right) \\ &= \sum_{\mu^{(l)}, \nu^{(l)} \in \{1, \dots, n\} \times l} \langle \widehat{\psi_{\mu^{(l)}}}, A \widehat{\psi_{\nu^{(l)}}} \rangle \varphi_{\mathbb{V}}\left(\left(\widehat{\omega_{\mu^{(l)}, \mathbb{V}}} \otimes E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}\right) e_{\mathbb{V}} \left(\widehat{\omega_{\nu^{(l)}, \mathbb{V}}} \otimes E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}\right)^*\right) \\ &= \sum_{\mu^{(l)}, \nu^{(l)} \in \{1, \dots, n\} \times l} \langle \widehat{\psi_{\mu^{(l)}}}, A \widehat{\psi_{\nu^{(l)}}} \rangle \varphi_{\omega_{\mathbb{V}}}(\widehat{\omega_{\mu^{(l)}, \mathbb{V}}} e_{\omega_{\mathbb{V}}}(\widehat{\omega_{\nu^{(l)}, \mathbb{V}}})^*) \\ &= \sum_{\mu^{(l)}, \nu^{(l)} \in \{1, \dots, n\} \times l} \langle \widehat{\psi_{\mu^{(l)}}}, A \widehat{\psi_{\nu^{(l)}}} \rangle \varphi_{\mathbf{u}}(\widehat{u_{\mu^{(l)}}} (\widehat{u_{\nu^{(l)}}})^*), \end{aligned}$$

$A \in \mathcal{A}_{[i, i+l-1]}, i \in \mathbb{Z}, l \in \mathbb{N}$ .

In the second equality we used the fact that the supports of  $e_{\mathbb{V}}, \varphi_{\mathbb{V}}$  are  $\widehat{P_R^{(n_0, \mathbb{V}, k_{R\mathbb{V}}, k_{L\mathbb{V}})}}, \widehat{P_L^{(n_0, \mathbb{V}, k_{R\mathbb{V}}, k_{L\mathbb{V}})}}$ .

In the third equality, we used the property  $\widehat{P_L^{(n_0, \mathbb{V}, k_{R\mathbb{V}}, k_{L\mathbb{V}})} \widehat{V_{\mu^{(l)}}} \widehat{P_R^{(n_0, \mathbb{V}, k_{R\mathbb{V}}, k_{L\mathbb{V}})}} = \widehat{\omega_{\mu^{(l)}, \mathbb{V}}} \otimes E_{00}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}$ . The eigenspace of  $T_{\mathbf{u}}$  for the eigenvalue 1 is  $\mathbb{C}\mathbb{I}$  and  $(M_{n_0, \mathbb{V}}, \mathbf{u}, \varphi_{\mathbf{u}})$  is a minimal standard triple which right generates  $\omega_{\mathbb{V}, \infty}$ . (Recall Appendix C of [O2].) By (34),  $\omega_{\mathbb{V}, \infty} = \omega_{\mathbb{A}, \infty}$  is right-generated by a minimal standard triple  $(M_1 = \mathbb{C}, \mathbf{v}, \text{id}_{\mathbb{C}})$ , where  $\mathbf{v} = (v_{\mu}) \in \mathbb{C}^{\times n}$  is given by  $v_1 = 1$  and  $v_{\mu} = 0$  for  $\mu \geq 2$ . From this and the uniqueness of the minimal representation proven in [FNW2] (see Theorem C.3 of [O2]), we conclude that there exists a  $C^*$ -isomorphism  $\Theta : M_{n_0, \mathbb{V}} \rightarrow \mathbb{C}$  such that  $v_{\mu} v_{\mu}^* = \Theta(u_{\mu} u_{\mu}^*)$  for all  $\mu = 1, \dots, n$ . Therefore,  $n_{0, \mathbb{V}} = 1$  and  $M_{n_0, \mathbb{V}} = \mathbb{C}$ .

Second, note that from the first and second equality of (36) and Lemma 3.13 of [O1], the map  $\Xi_{L, \mathbb{A}}^{-1} \circ \Xi_{L\mathbb{V}}$  (resp.  $\Xi_{R, \mathbb{A}}^{-1} \circ \Xi_{R\mathbb{V}}$ ) is a well-defined affine bijection from  $\mathcal{E}_{k_{L\mathbb{V}}+1}$  (resp.  $\mathcal{E}_{k_{R\mathbb{V}}+1}$ ) to  $\mathcal{E}_{n_0(k_L+1)}$  (resp.  $\mathcal{E}_{n_0(k_R+1)}$ ). Recall that  $\mathcal{E}_k$  denotes the set of all states on  $M_k$ . In particular, we have  $k_{R\mathbb{V}} + 1 = n_0(k_R + 1)$  and  $k_{L\mathbb{V}} + 1 = n_0(k_L + 1)$ .

Third, we show  $Y_{\mathbb{V}} = 0$ . We note that from the choice of  $\lambda_L, \lambda_R$ , the numbers

$$\{\overline{\lambda_{L\beta}} \lambda_{L\beta'} \overline{\lambda_{Rb}} \lambda_{Rb'} \mid \beta, \beta' = 0, \dots, n_0 - 1, b, b' = 0, \dots, k_L\} \quad (40)$$

are distinct. Furthermore,  $\overline{\lambda_{L\beta}} \lambda_{L\beta'} \overline{\lambda_{Rb}} \lambda_{Rb'}$  is a positive real number if and only if  $\beta = \beta'$  and  $b = b'$ .

From  $M_{n_0, \mathbb{V}} = \mathbb{C}$ , we have  $\omega_{1, \mathbb{V}} = u_1$  and it is an element in  $\mathbb{C}$  with absolute value 1. For  $\mu \geq 2$ , we have  $\omega_{\mu, \mathbb{V}} = 0$ . Note, in particular, that  $V_1$  is of the form

$$V_1 = \omega_1 \Lambda_{\lambda_{\mathbb{V}}} (\mathbb{I} + Y_{\mathbb{V}}) + W_1, \quad (41)$$

with some element  $W_1$ , such that

$$P_L^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})} W_1 = \sum_{\substack{i, j=0, \dots, k_{L\mathbb{V}} \\ |\lambda_{i, \mathbb{V}}| > |\lambda_{j, \mathbb{V}}|}} E_{ii}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})} W_1 E_{jj}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}, \quad (42)$$

(Recall that  $V \in M_{n_0V} \otimes \mathcal{D}(k_{RV}, k_{LV}, \mathbb{D}_V, \mathbb{G}_V) \Lambda_{\lambda_V} (1 + Y_V)$  from Definition 1.13 of [O1].) From (41) and (42), we have

$$\left( \sum_{i:i \geq 0, \lambda_{iV} = \lambda} E_{ii}^{(k_{RV}, k_{LV})} \right) V_1^N \left( \sum_{i:i \geq 0, \lambda_{iV} = \lambda} E_{ii}^{(k_{RV}, k_{LV})} \right) = (\omega_1 \lambda (\mathbb{I} + Y_V))^N \left( \sum_{i:i \geq 0, \lambda_{iV} = \lambda} E_{ii}^{(k_{RV}, k_{LV})} \right), \quad (43)$$

for any  $N \in \mathbb{N}$  and  $\lambda \in \{\lambda_{iV}\}_{i=1}^{k_{LV}}$ .

For each nonzero  $\xi \in \mathbb{C}^{k_{LV}+1}$ , we define a state  $\sigma_\xi$  on  $M_{k_{LV}+1}$  given by a density matrix

$$C_\xi^{-1} y_V^{-\frac{1}{2}} |\xi\rangle \langle \xi| y_V^{-\frac{1}{2}}$$

where  $C_\xi = \langle \xi, y_V^{-1} \xi \rangle$ . (Recall the definition of  $y_B$  in [O1], after Remark 2.10.) Furthermore, we denote by  $\tilde{\sigma}_\xi$  the image of  $\sigma_\xi$  by  $\Xi_{L,\mathbb{A}}^{-1} \circ \Xi_{LV}$ , i.e.,  $\tilde{\sigma}_\xi = \Xi_{L,\mathbb{A}}^{-1} \circ \Xi_{LV}(\sigma_\xi)$ . Let  $\tilde{D}_\xi$  be the density matrix of  $\tilde{\sigma}_\xi$ . We set  $D_\xi := y_{\mathbb{A}}^{\frac{1}{2}} \tilde{D}_\xi y_{\mathbb{A}}^{\frac{1}{2}} \in P_{L,\mathbb{A}}(M_{2n_0-1} \otimes M_{K_R+k_L+1}) P_{L,\mathbb{A}}$ . (Recall (24) for the definition of  $P_{L,\mathbb{A}}$ .) For each  $X \in M_{k_{LV}+1}$ , there exists an  $A_X \in \mathcal{A}_{(-\infty, -1]}^{\text{loc}}$  such that  $\mathbb{L}_V(A_X) = X$ . This can be seen from the proof of Lemma 3.11 of [O1]. From the definitions, we have

$$\begin{aligned} \sigma_\xi \left( y_V^{\frac{1}{2}} (\text{Ad } V_1^*)^N (\mathbb{L}_V(A_X)) y_V^{\frac{1}{2}} \right) &= \Xi_{LV}(\sigma_\xi) \left( A_X \otimes \left( e_{11}^{(n)} \right)^{\otimes N} \right) \\ &= \Xi_{L,\mathbb{A}}(\tilde{\sigma}_\xi) \left( A_X \otimes \left( e_{11}^{(n)} \right)^{\otimes N} \right) = \tilde{\sigma}_\xi \left( y_{\mathbb{A}}^{\frac{1}{2}} (\text{Ad } A_1^*)^N (\mathbb{L}_{\mathbb{A}}(A_X)) y_{\mathbb{A}}^{\frac{1}{2}} \right) \\ &= \sum_{\beta, \beta'=0}^{n_0-1} \sum_{b, b'=0}^{k_L} (\overline{\lambda_{L\beta}} \lambda_{L\beta'} \overline{\lambda_{Rb}} \lambda_{Rb'})^N \langle g_{\beta, b, \mathbb{A}}, \mathbb{L}_{\mathbb{A}}(A_X) g_{\beta', b', \mathbb{A}} \rangle \langle g_{\beta', b', \mathbb{A}}, D_\xi g_{\beta, b, \mathbb{A}} \rangle, \end{aligned} \quad (44)$$

for all  $N \in \mathbb{N}$  and  $X \in M_{k_{LV}+1}$ . (Recall (25) for the definition of  $g_{\beta, b, \mathbb{A}}$ .)

Let  $\lambda \in \{\lambda_{iV}\}_{i=1}^{k_{LV}}$ ,  $X \in \left( \sum_{i:i \geq 0, \lambda_{iV} = \lambda} E_{ii}^{(k_{RV}, k_{LV})} \right) M_{k_{RV}+k_{LV}+1} \left( \sum_{i:i \geq 0, \lambda_{iV} = \lambda} E_{ii}^{(k_{RV}, k_{LV})} \right)$ , and  $0 \neq \xi \in \left( \sum_{i:i \geq 0, \lambda_{iV} = \lambda} E_{ii}^{(k_{RV}, k_{LV})} \right) \mathbb{C}^{k_{RV}+k_{LV}+1}$ . Substituting this to (44) and using (43), we get

$$\begin{aligned} C_\xi^{-1} \sum_{k=0}^{2(k_{LV}+1)} |\lambda|^{2N} {}_N C_k \left\langle \xi, (\text{Ad}(1 + Y_V^*) - \mathbb{I})^k (X) \xi \right\rangle &= C_\xi^{-1} |\lambda|^{2N} \left\langle \xi, (\text{Ad}(1 + Y_V^*))^N (X) \xi \right\rangle \\ &= \sigma_\xi \left( y_V^{\frac{1}{2}} (\text{Ad } V_1^*)^N (X) y_V^{\frac{1}{2}} \right) = \sigma_\xi \left( y_V^{\frac{1}{2}} (\text{Ad } V_1^*)^N (\mathbb{L}_V(A_X)) y_V^{\frac{1}{2}} \right) \\ &= \sum_{\beta, \beta'=0}^{n_0-1} \sum_{b, b'=0}^{k_L} (\overline{\lambda_{L\beta}} \lambda_{L\beta'} \overline{\lambda_{Rb}} \lambda_{Rb'})^N \langle g_{\beta, b, \mathbb{A}}, \mathbb{L}_{\mathbb{A}}(A_X) g_{\beta', b', \mathbb{A}} \rangle \langle g_{\beta', b', \mathbb{A}}, D_\xi g_{\beta, b, \mathbb{A}} \rangle, \end{aligned} \quad (45)$$

for all  $N \in \mathbb{N}$ . Applying Lemma C.7 of [O1], we see that  $\left\langle \xi, (\text{Ad}(1 + Y_V^*) - \mathbb{I})^k (X) \xi \right\rangle = 0$  for all  $k \in \mathbb{N}$ . As this holds for all  $\lambda, \xi$  and  $X$  as above, we conclude that  $Y_V = 0$ .

Finally, we show that  $\lambda_V \in \text{Wo}'(n_0(k_R+1)-1, n_0(k_L+1)-1)$ . Namely, we show that the numbers  $\{\lambda_{i,V}\}_{i=1}^{k_{LV}}$  (resp.  $\{\lambda_{i,V}\}_{i=k_{RV}+1}^{-1}$ ) are distinct. Substituting  $Y_V = 0$  to (45), we have

$$C_\xi^{-1} |\lambda|^{2N} \langle \xi, X \xi \rangle = \sum_{\beta, \beta'=0}^{n_0-1} \sum_{b, b'=0}^{k_L} (\overline{\lambda_{L\beta}} \lambda_{L\beta'} \overline{\lambda_{Rb}} \lambda_{Rb'})^N \langle g_{\beta, b, \mathbb{A}}, \mathbb{L}_{\mathbb{A}}(A_X) g_{\beta', b', \mathbb{A}} \rangle \langle g_{\beta', b', \mathbb{A}}, D_\xi g_{\beta, b, \mathbb{A}} \rangle, \quad (46)$$

for all  $N \in \mathbb{N}$ ,  $\lambda \in \{\lambda_{i\mathbb{V}}\}_{i=1}^{k_{L\mathbb{V}}}$ ,  $X \in \left(\sum_{i:i \geq 0, \lambda_{i\mathbb{V}}=\lambda} E_{ii}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}\right) \mathbb{M}_{k_{R\mathbb{V}}+k_{L\mathbb{V}}+1} \left(\sum_{i:i \geq 0, \lambda_{i\mathbb{V}}=\lambda} E_{ii}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}\right)$ , and  $0 \neq \xi \in \left(\sum_{i:i \geq 0, \lambda_{i\mathbb{V}}=\lambda} E_{ii}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}\right) \mathbb{C}^{k_{R\mathbb{V}}+k_{L\mathbb{V}}+1}$ . In particular, for each  $\lambda \in \{\lambda_{i\mathbb{V}}\}_{i=1}^{k_{L\mathbb{V}}}$ , take any unit vector  $\xi \in \sum_{i:i \geq 0, \lambda_{i\mathbb{V}}=\lambda} E_{ii}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})} \mathbb{C}^{k_{R\mathbb{V}}+k_{L\mathbb{V}}+1}$  and set  $X := |\xi\rangle \langle \xi|$ . Then we obtain

$$C_\xi^{-1} |\lambda|^{2N} = \sum_{\beta, \beta'=0}^{n_0-1} \sum_{b, b'=0}^{k_L} (\overline{\lambda_{L\beta}} \lambda_{L\beta'} \overline{\lambda_{Rb}} \lambda_{Rb'})^N \langle g_{\beta, b, \mathbb{A}}, \mathbb{L}_{\mathbb{A}} (A_{|\xi\rangle \langle \xi|}) g_{\beta', b', \mathbb{A}} \rangle \langle g_{\beta', b', \mathbb{A}}, D_\xi g_{\beta, b, \mathbb{A}} \rangle,$$

for all  $N \in \mathbb{N}$ . As the coefficient  $C_\xi^{-1}$  of  $|\lambda|^{2N}$  on the left hand side is nonzero, another application of Lemma C.7 of [O1] and distinction of numbers in (40) implies that for each  $\lambda \in \{\lambda_{i\mathbb{V}}\}_{i=1}^{k_{L\mathbb{V}}}$ , there exists  $\overline{\lambda_{L\beta}} \lambda_{L\beta'} \overline{\lambda_{Rb}} \lambda_{Rb'}$  such that  $|\lambda|^2 = \overline{\lambda_{L\beta}} \lambda_{L\beta'} \overline{\lambda_{Rb}} \lambda_{Rb'}$ . By the consideration around (40), we see that  $\beta = \beta'$ ,  $b = b'$ ,  $|\lambda|^2 = |\lambda_{L\beta}|^2 |\lambda_{Rb}|^2$ . Furthermore, as the numbers in (40) are distinct, such a pair  $(\beta, b)$  is unique. We denote the pair by  $(\beta_\lambda, b_\lambda)$ . For any unit vector  $\xi \in \sum_{i:i \geq 0, \lambda_{i\mathbb{V}}=\lambda} E_{ii}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}$ , we have

$$\langle g_{\beta_\lambda, b_\lambda, \mathbb{A}}, \mathbb{L}_{\mathbb{A}} (A_{|\xi\rangle \langle \xi|}) g_{\beta_\lambda, b_\lambda, \mathbb{A}} \rangle \langle g_{\beta_\lambda, b_\lambda, \mathbb{A}}, D_\xi g_{\beta_\lambda, b_\lambda, \mathbb{A}} \rangle = C_\xi^{-1} \neq 0. \quad (47)$$

Now we are ready to show that  $\lambda_{i\mathbb{V}} \neq \lambda_{j\mathbb{V}}$  for any  $1 \leq i < j \leq k_{L\mathbb{V}}$ . This proves that the numbers  $\{\lambda_{i\mathbb{V}}\}_{i=1}^{k_{L\mathbb{V}}}$  are distinct. We assume that  $\lambda_{i\mathbb{V}} = \lambda_{j\mathbb{V}}$  for some  $1 \leq i < j \leq k_{L\mathbb{V}}$  and show a contradiction. First, substituting  $\lambda := \lambda_{i\mathbb{V}} = \lambda_{j\mathbb{V}}$ ,  $X = E_{j,j}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}$  and  $\xi = f_i^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}$  to (46), we have

$$0 = \sum_{\beta, \beta'=0}^{n_0-1} \sum_{b, b'=0}^{k_L} (\overline{\lambda_{L\beta}} \lambda_{L\beta'} \overline{\lambda_{Rb}} \lambda_{Rb'})^N \left\langle g_{\beta, b, \mathbb{A}}, \mathbb{L}_{\mathbb{A}} \left( A_{E_{j,j}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}} \right) g_{\beta', b', \mathbb{A}} \right\rangle \left\langle g_{\beta', b', \mathbb{A}}, D_{f_i^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}} g_{\beta, b, \mathbb{A}} \right\rangle,$$

for all  $N \in \mathbb{N}$ . As the numbers in (40) are distinct, this equality implies

$$\left\langle g_{\beta, b, \mathbb{A}}, \mathbb{L}_{\mathbb{A}} \left( A_{E_{j,j}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}} \right) g_{\beta', b', \mathbb{A}} \right\rangle \left\langle g_{\beta', b', \mathbb{A}}, D_{f_i^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}} g_{\beta, b, \mathbb{A}} \right\rangle = 0, \quad (48)$$

for all  $\beta, \beta' = 0, \dots, n_0 - 1$  and  $b, b' = 0, \dots, k_L$ . For  $\lambda = \lambda_{i\mathbb{V}} = \lambda_{j\mathbb{V}}$ , there exists a unique pair  $(\beta_\lambda, b_\lambda)$  such that  $|\lambda| = |\lambda_{L\beta_\lambda} \lambda_{Rb_\lambda}|$ . From (47) with  $\xi = f_i^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}$ , and  $\xi = f_j^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}$ , we have

$$\begin{aligned} \left\langle g_{\beta_\lambda, b_\lambda, \mathbb{A}}, \mathbb{L}_{\mathbb{A}} \left( A_{E_{ii}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}} \right) g_{\beta_\lambda, b_\lambda, \mathbb{A}} \right\rangle \left\langle g_{\beta_\lambda, b_\lambda, \mathbb{A}}, D_{f_i^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}} g_{\beta_\lambda, b_\lambda, \mathbb{A}} \right\rangle &\neq 0, \\ \left\langle g_{\beta_\lambda, b_\lambda, \mathbb{A}}, \mathbb{L}_{\mathbb{A}} \left( A_{E_{jj}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}} \right) g_{\beta_\lambda, b_\lambda, \mathbb{A}} \right\rangle \left\langle g_{\beta_\lambda, b_\lambda, \mathbb{A}}, D_{f_j^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}} g_{\beta_\lambda, b_\lambda, \mathbb{A}} \right\rangle &\neq 0. \end{aligned} \quad (49)$$

On the other hand, we have

$$\left\langle g_{\beta_\lambda, b_\lambda, \mathbb{A}}, \mathbb{L}_{\mathbb{A}} \left( A_{E_{jj}^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}} \right) g_{\beta_\lambda, b_\lambda, \mathbb{A}} \right\rangle \left\langle g_{\beta_\lambda, b_\lambda, \mathbb{A}}, D_{f_i^{(k_{R\mathbb{V}}, k_{L\mathbb{V}})}} g_{\beta_\lambda, b_\lambda, \mathbb{A}} \right\rangle = 0,$$

from (48). This and (49) can not hold simultaneously. Hence we obtain a contradiction. Similarly, we can show that the numbers  $\{\lambda_{i\mathbb{V}}\}_{i=1}^{k_{L\mathbb{V}}}$  are distinct.  $\square$

#### 4.4 Properties of $\mathcal{D}_l$

Throughout this subsection, let  $2 \leq n_0 \in \mathbb{N}$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , and  $\mathbb{A}, \mathbb{B}(t)$  be given by (23), (22). Recall that  $\mathcal{D}_l$  is the linear subspace of  $\bigotimes_{i=0}^{l-1} \mathbb{C}^n$  spanned by  $\xi_{\alpha, \beta, a, b}^{(l)}$ ,  $\alpha, \beta \in \{1, \dots, n_0\}$ ,  $a = 0, 1, \dots, k_R$ ,  $b = 0, 1, \dots, k_L$ , and that  $G_l$  is the orthogonal projection onto  $\mathcal{D}_l$ . In this subsection, we investigate properties of  $\mathcal{D}_l$ . First we show that the projections  $G_l$  and  $G_{l, \mathbb{A}}$  are close. In order for that, we estimate  $\Upsilon_l$  in (27).

**Lemma 4.12.** *There exists a constant  $C_7 > 0$  such that*

$$\|r_\alpha^l \Upsilon_l(x)\| \leq C_7 \kappa^{\frac{l}{2}} \|x\|,$$

we for all  $\alpha = 2, \dots, n_0$ ,  $l \in \mathbb{N}$ , and  $x \in \mathbb{M}_{k_R+k_L+1}$ .

**Proof.** From (27), it suffices to show that each  $\zeta_{i_1 \dots i_k}^{(k)l}(e_{11}^{(n_0)} \otimes x)$ , is bounded by  $\text{const} \times l^k \times \|x\|$ . By the definition

$$\begin{aligned} & \left\| \zeta_{i_1 \dots i_k}^{(k)l} \left( e_{11}^{(n_0)} \otimes x \right) \right\|^2 \\ &= \sum_{1 \leq m_1 < \dots < m_k \leq l} \sum_{j, j' = -k_R}^{k_L} \left\langle \chi_1^{(n_0)} \otimes f_j^{(k_R, k_L)}, T_0^{m_1-1} \circ \text{Ad } K_{i_1} \circ \dots \circ T_0^{m_k-m_{k-1}-1} \circ \text{Ad } K_{i_k} \circ T_0^{l-m_k} \left( e_{11}^{(n_0)} \otimes x^* E_{jj'}^{(k_R, k_L)} x \right) \left( \chi_1^{(n_0)} \otimes f_{j'}^{(k_R, k_L)} \right) \right\rangle. \end{aligned}$$

Here,  $T_0$  is a CP map given by  $T_0 := \text{Ad}(R \otimes \Lambda_{\lambda_R})$ , and we have a bound  $\|T_0\| = \|T_0(1)\| \leq 1$ . From this and the above formula, we obtain the bound

$$\left\| \zeta_{i_1 \dots i_k}^{(k)l} \left( e_{11}^{(n_0)} \otimes x \right) \right\|^2 \leq \sum_{1 \leq m_1 < \dots < m_k \leq l} \sum_{j, j' = -k_R}^{k_L} \left( \max_{i=1,2,3} \{\|\text{Ad } K_i\|\} \right)^k \|x\|^2 \leq \text{const} \cdot l^k \cdot \|x\|^2.$$

This completes the proof.  $\square$

**Lemma 4.13.** *There exist  $l_3 \in \mathbb{N}$  and a constant  $C_8 > 0$  such that for all  $l \geq l_3$ ,*

1. *the vectors  $\xi_{\alpha, \beta, a, b}^{(l)}$ ,  $\alpha, \beta \in \{1, \dots, n_0\}$ ,  $a = 0, 1, \dots, k_R$ , and  $b = 0, 1, \dots, k_L$ , are linearly independent, and*
- 2.

$$\|G_l - G_{l, \mathbb{A}}\| \leq C_8 \kappa^{\frac{l}{2}}.$$

**Proof.** This follows from the fact that there exist  $C' > 0$  and  $L_{\mathbb{A}} \in \mathbb{N}$  such that

$$C' \|X\|_2 \leq \left\| \Gamma_{l, \mathbb{A}}^{(R)}(X) \right\| \leq (C')^{-1} \|X\|_2, \quad \text{for all } L_{\mathbb{A}} \leq l, \quad X \in P_{R, \mathbb{A}}(\mathbb{M}_{n_0} \otimes \mathbb{M}_{k_R+k_L+1}) P_{L, \mathbb{A}} \quad (50)$$

and Lemma 4.12. (Recall Lemma 2.16 of [O1].)  $\square$

**Lemma 4.14.** *For any  $l \geq \max\{l_3, n_0^6(k_R+1)(k_L+1)\}$  (where  $l_3$  is given in Lemma 4.13),  $t \in [0, 1]$ , and  $\alpha, \beta \in \{1, \dots, n_0\}$ ,  $a = 0, 1, \dots, k_R$ , and  $b = 0, 1, \dots, k_L$ , set*

$$\hat{\xi}_{\alpha, \beta, a, b}^{(l)}(t) := \begin{cases} \Gamma_{l, \mathbb{B}(t)}^{(R)} \left( \Theta_{l, t} \left( e_{\alpha\beta}^{(n_0)} \otimes E_{-a, b}^{(k_R, k_L)} \right) \right), & \text{if } t \in (0, 1] \\ \xi_{\alpha, \beta, a, b}^{(l)}, & \text{if } t = 0 \end{cases}. \quad (51)$$

Let  $\hat{G}_l(t)$  be the orthogonal projection onto the subspace of  $\bigotimes_{i=0}^{l-1} \mathbb{C}^n$  spanned by  $\hat{\xi}_{\alpha, \beta, a, b}^{(l)}(t)$ ,  $\alpha, \beta \in \{1, \dots, n_0\}$ ,  $a = 0, 1, \dots, k_R$ , and  $b = 0, 1, \dots, k_L$ , for each  $t \in [0, 1]$  and  $l \geq \max\{l_3, n_0^6(k_R+1)(k_L+1)\}$ . Then the maps  $[0, 1] \ni t \mapsto \hat{\xi}_{\alpha, \beta, a, b}^{(l)}(t) \in \bigotimes_{i=0}^{l-1} \mathbb{C}^n$ ,  $[0, 1] \ni t \mapsto \hat{G}_l(t) \in \bigotimes_{i=0}^{l-1} \mathbb{M}_n$  are  $C^\infty$ .

**Proof.** That  $[0, 1] \ni t \mapsto \hat{\xi}_{\alpha, \beta, a, b}^{(l)}(t) \in \bigotimes_{i=0}^{l-1} \mathbb{C}^n$  is  $C^\infty$ , is immediate from the proof of Lemma 4.4. By Lemma 4.13, the vectors  $\xi_{\alpha, \beta, a, b}^{(l)}$  are linearly independent for any  $l \geq l_3$ . For any  $t \in (0, 1]$ ,  $\mathbb{B}(t)$  belongs to  $\text{Class}(n, n_0, k_R, k_L) \subset \text{ClassA}$  from Lemma 4.1. Therefore, by Proposition 3.1 of [O1] and Lemma 3.5, for any  $l \geq n_0^6(k_R+1)(k_L+1) \geq l_{\mathbb{B}(t)}$ , the vectors  $\left\{ \Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l, t} \left( e_{\alpha\beta}^{(n_0)} \otimes E_{-a, b}^{(k_R, k_L)} \right) \right\}_{\alpha, \beta=1, \dots, n_0, a=0, \dots, k_R, b=0, \dots, k_L}$  are linearly independent. Therefore, by Lemma A.1,  $\hat{G}_l$  is  $C^\infty$  for  $l \geq \max\{l_3, n_0^6(k_R+1)(k_L+1)\}$ .  $\square$

By the bijectivity of  $\Theta_{l, t}$  on  $M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$  and the definition of  $G_l$ , we have  $\hat{G}_l(t) = G_{l, \mathbb{B}(t)}$  for  $t \in (0, 1]$  and  $\hat{G}_l(0) = G_l$ .

Next we show the intersection property of  $\mathcal{D}_l$ . It is convenient to represent the vectors  $\xi_{\alpha, \beta, a, b}^{(l)}$  in a matrix product form. We set  $\mathbb{F} \in (M_{n_0+1} \otimes M_{k_R+k_L+1})^{\times n}$  by

$$\begin{aligned} F_1 &:= \Lambda_{\lambda_F} \otimes \Lambda_{\lambda_R}, \\ F_2 &:= \left| \chi_1^{(n_0+1)} \right\rangle \langle \eta_F | \otimes \Lambda_{\lambda_R} + |\eta_F\rangle \left\langle \chi_{n_0+1}^{(n_0+1)} \right| \otimes \Lambda_{\lambda_R} + \Lambda_{\lambda_F} \otimes V \Lambda_{\lambda_R}, \\ F_\mu &= 0, \quad \mu \geq 3. \end{aligned}$$

Here, we set

$$\eta_F := \sum_{i=2}^{n_0} \chi_i^{(n_0+1)} \in \mathbb{C}^{n_0+1}, \quad \lambda_F := (1, r_2, \dots, r_{n_0}, 1) \in \mathbb{C}^{n_0+1}.$$

(Recall  $r_\alpha, V$  from (19), and (20).) With this definition, it is straight forward to check

$$\Upsilon_l(X) = \Gamma_{l, \mathbb{F}}^{(R)} \left( e_{1, n_0+1}^{(n_0+1)} \otimes X \right),$$

for  $X \in P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ ,  $l \in \mathbb{N}$ . In order to represent  $\xi_{\alpha, \beta, a, b}^{(l)}$ , we set  $\mathbb{A}'$  as

$$A'_\mu := A_\mu \oplus F_\mu \in (M_{2n_0-1} \otimes M_{k_R+k_L+1}) \oplus (M_{n_0+1} \otimes M_{k_R+k_L+1}) \simeq (M_{2n_0-1} \oplus M_{n_0+1}) \otimes M_{k_R+k_L+1}, \quad \mu = 1, \dots, n. \quad (52)$$

Using these matrices, our  $\xi_{\alpha, \beta, a, b}^{(l)}$  is represented as

$$\xi_{\alpha, \beta, a, b}^{(l)} = \Gamma_{l, \mathbb{A}'}^{(R)} \left( \Theta_l \left( e_{\alpha\beta}^{(n_0)} \right) \otimes E_{-a, b}^{(k_R, k_L)} \right).$$

Here,  $\Theta_l : M_{n_0} \rightarrow M_{2n_0-1} \oplus M_{n_0+1}$  is a linear map given by

$$\Theta_l \left( e_{\alpha\beta}^{(n_0)} \right) := E_{-(\alpha-1), \beta-1}^{(n_0-1, n_0-1)} \oplus \left( -\delta_{\alpha\beta} (1 - \delta_{\alpha, 1}) \tilde{r}_\alpha^l e_{1, n_0+1}^{(n_0+1)} \right), \quad \alpha, \beta = 1, \dots, n_0. \quad (53)$$

Hence, we obtain

$$\mathcal{D}_l = \Gamma_{l, \mathbb{A}'}^{(R)} \left( (\Theta_l \otimes \text{id}) \left( M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)} \right) \right), \quad (54)$$

and

$$\lim_{t \downarrow 0} \Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l, t}(X) = \Gamma_{l, \mathbb{A}'}^{(R)} \circ (\Theta_l \otimes \text{id})(X), \quad X \in M_{n_0} \otimes M_{k_R+k_L+1}, \quad l \in \mathbb{N}. \quad (55)$$

**Lemma 4.15.** *There exists an  $l_4 \in \mathbb{N}$  such that*

$$\mathcal{D}_{l+1} = (\mathcal{D}_l \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes \mathcal{D}_l), \quad l_4 \leq l.$$

**Proof.** The inclusion  $\subset$  is easy. For each  $l \geq \max\{2n_0^6(k_R + 1)(k_L + 1), l_3\}$  and  $t \in (0, 1]$ , by Proposition 3.1 of [O1] and Lemma 3.5, Lemma 4.1, we have  $l \geq 2n_0^6(k_R + 1)(k_L + 1) \geq 2l_{\mathbb{B}(t)} \geq m_{\mathbb{B}(t)}$ . Therefore, we have

$$\mathcal{G}_{l+1, \mathbb{B}(t)} = (\mathcal{G}_{l, \mathbb{B}(t)} \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes \mathcal{G}_{l, \mathbb{B}(t)}).$$

For any  $\eta \in \mathcal{D}_{l+1}$ , by Lemma 4.14, we have

$$\eta = G_{l+1} \eta = \lim_{t \downarrow 0} G_{l+1, \mathbb{B}(t)} \eta = \begin{cases} \lim_{t \downarrow 0} (G_{l, \mathbb{B}(t)} \otimes \mathbb{I}) G_{l+1, \mathbb{B}(t)} \eta = (G_l \otimes \mathbb{I}) \eta \\ \lim_{t \downarrow 0} (\mathbb{I} \otimes G_{l, \mathbb{B}(t)}) G_{l+1, \mathbb{B}(t)} \eta = (\mathbb{I} \otimes G_l) \eta. \end{cases}$$

This proves  $\mathcal{D}_{l+1} \subset (\mathcal{D}_l \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes \mathcal{D}_l)$ .

The proof of opposite inclusion is similar to the proof of Lemma 2.14 of [O1].

By the routine argument using Lemma C.7 of [O1], we see that there exists an  $l'_4 \in \mathbb{N}$  such that

$$\mathcal{K}_l(\mathbb{A}') = \mathcal{V}_l,$$

for all  $l \geq l'_4$ . The subspace  $\mathcal{V}_l$  is defined by

$$\mathcal{V}_l = \text{span} \left( \left\{ \begin{aligned} & \left( \mathbb{I}_{2n_0-1} \oplus \mathbb{I}_{n_0+1} \right), \left( E_{0, \beta-1}^{(n_0-1, n_0-1)} \oplus e_{1\beta}^{(n_0+1)} \right), \\ & \left( E_{-(\alpha-1), 0}^{(n_0-1, n_0-1)} \oplus e_{\alpha, n_0+1}^{(n_0+1)} \right), \\ & \left( 0_{2n_0-1} \oplus e_{1, n_0+1}^{(n_0+1)} \right), \\ & \left( E_{-(\alpha-1), \beta-1}^{(n_0-1, n_0-1)} \oplus 0_{n_0+1} \right) \end{aligned} \right\}_{\alpha, \beta=2, \dots, n_0} \otimes \left\{ \begin{aligned} & \mathbb{I}, E_{-a, 0}^{(k_R, k_L)}, \\ & E_{0, b}^{(k_R, k_L)}, E_{-a, b}^{(k_R, k_L)} \end{aligned} \right\}_{a=1, \dots, k_R, b=1, \dots, k_L} \right) \hat{\Lambda}^l. \quad (56)$$

Here, we set  $\hat{\Lambda} := (\Lambda_{\lambda_L} \oplus \Lambda_{\lambda_F}) \otimes \Lambda_{\lambda_R}$ .

Now we are ready to prove  $\mathcal{D}_{l+1} = (\mathcal{D}_l \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes \mathcal{D}_l)$  for  $l \geq l'_4 + 1$ . Let  $l'_4 + 1 \leq l \in \mathbb{N}$  and  $\Phi \in (\mathcal{D}_l \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes \mathcal{D}_l)$ . We show that  $\Phi \in \mathcal{D}_{l+1}$ . By (54), there exist  $C_\mu, D_\nu \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ ,  $\mu, \nu = 1, \dots, n$  such that

$$\Phi = \sum_{\mu=1}^n \Gamma_{l, \mathbb{A}'}^{(R)}((\Theta_l \otimes \text{id})(C_\mu)) \otimes \psi_\mu = \sum_{\nu=1}^n \psi_\nu \otimes \Gamma_{l, \mathbb{A}'}^{(R)}((\Theta_l \otimes \text{id})(D_\nu)). \quad (57)$$

Therefore, for all  $\varpi^{(l-1)} \in \{1, \dots, n\}^{\times l-1}$  and  $\mu, \nu \in \{1, \dots, n\}$ , we have

$$\text{Tr} \left( (\Theta_l \otimes \text{id})(C_\mu) \left( A'_{\nu} \widehat{A'_{\varpi^{(l-1)}}} \right)^* \right) = \text{Tr} \left( (\Theta_l \otimes \text{id})(D_\nu) \left( \widehat{A'_{\varpi^{(l-1)}}} A'_\mu \right)^* \right).$$

From this, we see that for each  $\mu, \nu \in \{1, \dots, n\}$ ,

$$A'^*_{\nu} (\Theta_l \otimes \text{id})(C_\mu) - (\Theta_l \otimes \text{id})(D_\nu) A'^*_{\mu}$$

is orthogonal to  $\mathcal{K}_{l-1}(\mathbb{A}')$  with respect to the inner product  $\langle \cdot, \cdot \rangle_{\text{Tr}}$  given by  $\langle X, Y \rangle_{\text{Tr}} = \text{Tr} X^* Y$ . As  $l-1 \geq l'_4$ , we have  $\mathcal{K}_{l-1}(\mathbb{A}') = \mathcal{V}_{l-1}$ .

Note that  $A'_1{}^{-1} \mathcal{V}_l \subset \mathcal{V}_{l-1}$  from the definition of  $\mathcal{V}_l$ . From this, if  $X \in (M_{2n_0-1} \oplus M_{n_0+1}) \otimes M_{k_R+k_L+1}$  belongs to the orthogonal complement of  $\mathcal{V}_{l-1}$  with respect to  $\langle \cdot, \cdot \rangle_{\text{Tr}}$ , then  $(A'_1{}^*)^{-1} X$  belongs to the orthogonal complement of  $\mathcal{V}_l$ . Applying this to  $X = A'_1{}^* (\Theta_l \otimes \text{id})(C_\mu) - (\Theta_l \otimes \text{id})(D_1) A'^*_{\mu}$ , we see for each  $\mu = 1, \dots, n$ , that

$$Z_\mu := (\Theta_l \otimes \text{id})(C_\mu) - (A'_1{}^*)^{-1} (\Theta_l \otimes \text{id})(D_1) A'^*_{\mu}$$

is orthogonal to  $\mathcal{V}_l = \mathcal{K}_l(\mathbb{A}')$ . This means that  $Z_\mu$  is in the kernel of  $\Gamma_{l,\mathbb{A}'}^{(R)}$  for each  $\mu = 1, \dots, n$ .

By the definition of  $\Theta_l$ , we have

$$(A_1'^*)^{-1} \left( (\Theta_l \otimes \text{id}) \left( e_{\alpha\beta}^{(n_0)} \otimes E_{-a,b}^{(k_R,k_L)} \right) \right) = (\Theta_{l+1} \otimes \text{id}) \left( \left( \overline{\lambda_{L,-(\alpha-1)}} \right)^{-1} \left( \overline{\lambda_{R,-a}} \right)^{-1} e_{\alpha\beta}^{(n_0)} \otimes E_{-a,b}^{(k_R,k_L)} \right).$$

Therefore, we have

$$(A_1'^*)^{-1} (\Theta_l \otimes \text{id}) \left( M_{n_0} \otimes P_R^{(k_R,k_L)} M_{k_R+k_L+1} P_L^{(k_R,k_L)} \right) \subset (\Theta_{l+1} \otimes \text{id}) \left( M_{n_0} \otimes P_R^{(k_R,k_L)} M_{k_R+k_L+1} P_L^{(k_R,k_L)} \right).$$

In particular, we have

$$(A_1'^*)^{-1} (\Theta_l \otimes \text{id}) (D_1) = (\Theta_{l+1} \otimes \text{id}) (W),$$

with some  $W \in M_{n_0} \otimes P_R^{(k_R,k_L)} M_{k_R+k_L+1} P_L^{(k_R,k_L)}$ . Hence for each  $\mu = 1, \dots, n$ , we obtain

$$(\Theta_l \otimes \text{id}) (C_\mu) = (\Theta_{l+1} \otimes \text{id}) (W) A_\mu'^* + Z_\mu,$$

where  $W \in M_{n_0} \otimes P_R^{(k_R,k_L)} M_{k_R+k_L+1} P_L^{(k_R,k_L)}$  and  $Z_\mu \in \ker \Gamma_{l,\mathbb{A}'}^{(R)}$ . Substituting this to (57), we obtain

$$\Phi = \sum_{\mu=1}^n \Gamma_{l,\mathbb{A}'}^{(R)} ((\Theta_l \otimes \text{id}) (C_\mu)) \otimes \psi_\mu = \Gamma_{l+1,\mathbb{A}'}^{(R)} ((\Theta_{l+1} \otimes \text{id}) (W)) \in \mathcal{D}_{l+1}.$$

This completes the proof.  $\square$

The Hamiltonian  $H_{\Phi_{1-G_m}}$  given by the interaction  $1 - G_m$  via the formula (1), (2) is gapped with respect to the open boundary conditions:

**Lemma 4.16.** *For any  $m, N \in \mathbb{N}$  with  $\max\{l_3, l_4\} \leq m \leq N$ , the kernel of  $(H_{\Phi_{1-G_m}})_{[0,N-1]}$  is equal to  $\mathcal{D}_N$ , and its dimension is  $n_0^2(k_L + 1)(k_R + 1)$ . For each  $m \geq \max\{l_3, l_4\}$ , there exist a  $\gamma_m > 0$  and an  $N_m \in \mathbb{N}$  such that*

$$\gamma_m (1 - G_N) \leq (H_{\Phi_{1-G_m}})_{[0,N-1]}, \quad \text{for all } N \geq N_m.$$

(The numbers  $l_3, l_4$  are given in Lemma 4.13, Lemma 4.15.)

**Proof.** From Lemma 4.15,  $\{\mathcal{D}_l\}_l$  satisfies the intersection property. From Lemma 4.6,  $\{\mathcal{G}_{l,\mathbb{A}}\}_l$  satisfies *Condition 1* of [O1] Definition 2.1. By the second statement of Lemma 4.13, the difference between  $G_l$  and  $G_{l,\mathbb{A}}$  decays exponentially fast, as  $l \rightarrow \infty$ . Therefore,  $\{\mathcal{D}_l\}_l$  satisfies the second property of *Condition 1*. Hence we conclude that  $\{\mathcal{D}_l\}_l$  satisfies *Condition 1*. Therefore, by the Theorem of Nachtergaele, [N] (see Theorem 2.2 of [O1] for the version we use) we obtain the claim of the current Lemma.  $\square$

Note that for  $m \geq \max\{l_3, l_4\}$ , we have  $n^m > n_0^2(k_R + 1)(k_L + 1)$  by Remark 2.2. The Hamiltonian  $H_{\Phi_{1-G_{m_1}}}$  and  $H_{\Phi_{m_2,\mathbb{A}}}$  are equivalent with respect to the typeII- $C^1$ -classification.

**Lemma 4.17.** *Let  $m_1, m_2 \in \mathbb{N}$  with  $\max\{l_3, l_4\} \leq m_1$  and  $\max\{l_{\mathbb{A}}, m_{\mathbb{A}}\} \leq m_2$ . (Recall the definition of  $l_{\mathbb{A}}$  from Lemma 4.5.) For each  $t \in [0, 1]$  we define  $\Phi(t) \in \mathcal{J}$  by*

$$\Phi(X; t) := (1 - t)\Phi_{1-G_{m_1}}(X) + t\Phi_{m_2,\mathbb{A}}(X), \quad X \in \mathfrak{S}_{\mathbb{Z}}.$$

Then we have  $H_{\Phi_{1-G_{m_1}}} \simeq_{II} H_{\Phi_{m_2,\mathbb{A}}}$ .

**Proof.** It is proven by the same argument as in the proof of Corollary 1.4 of [O2].  $\square$

At the end of this subsection, we prove Lemmas which we will use in the next section to discuss about the bulk classification. We introduce the following notation.

**Notation 4.18.** Let  $H_\Phi$  be a frustration free Hamiltonian given by a positive interaction  $\Phi \in \mathcal{J}$ . For each  $\Gamma \subset \mathbb{Z}$ , we set

$$\widetilde{\mathcal{S}}_{\mathbb{Z},\Gamma}(H_\Phi) := \{\varphi \mid \varphi \text{ is a state on } \mathcal{A}_{\mathbb{Z}} \text{ with } \varphi(\Phi(X)) = 0 \text{ for all } X \in \mathfrak{S}_\Gamma\}.$$

*Remark 4.19.* By definition, we clearly have  $\widetilde{\mathcal{S}}_{\mathbb{Z},\Gamma_1}(H_\Phi) \subset \widetilde{\mathcal{S}}_{\mathbb{Z},\Gamma_2}(H_\Phi)$  if  $\Gamma_2 \subset \Gamma_1$ .

From 2. of Lemma 4.13, we obtain the following.

**Lemma 4.20.** For any  $m_1, m_2 \in \mathbb{N}$  with  $m_1 \geq l_4$ ,  $m_2 \geq m_\mathbb{A}$ , and  $M \in \mathbb{N}$ , we have

$$\mathcal{S}_{\mathbb{Z}}(H_{\Phi_{1-G_{m_1}}}) = \mathcal{S}_{\mathbb{Z}}(H_{\Phi_{m_2,\mathbb{A}}}), \quad \widetilde{\mathcal{S}_{\mathbb{Z},[-M,M]^c}(H_{\Phi_{1-G_{m_1}}})} = \widetilde{\mathcal{S}_{\mathbb{Z},[-M,M]^c}(H_{\Phi_{m_2,\mathbb{A}}})}. \quad (58)$$

**Definition 4.21.** We say that an interaction  $\Phi \in \mathcal{J}$  satisfies *Condition 6* if the followings hold.

1. There exists a state  $\omega_\infty$  on  $\mathcal{A}_{\mathbb{Z}}$  such that  $\mathcal{S}_{\mathbb{Z}}(H_\Phi) = \{\omega_\infty\}$ .
2. For any  $M \in \mathbb{N}$  and  $\varphi \in \widetilde{\mathcal{S}_{\mathbb{Z},[-M,M]^c}(H_\Phi)}$ ,  $\varphi$  is quasi-equivalent to  $\omega_\infty$ .

**Lemma 4.22.** Let  $h \in \mathcal{A}_{\mathbb{Z}}$  be a positive operator given by either of the followings:

$$h := \begin{cases} 1 - G_{m,\mathbb{B}}, & \mathbb{B} \in \text{Class A}, \quad m \geq m_\mathbb{B}, \\ 1 - G_{m,\mathbb{A}}, & \mathbb{A} : \text{given in (23)}, \quad m \geq m_\mathbb{A}. \\ 1 - G_m, & G_m : \text{given in (29)}, \quad m \geq l_4 \end{cases}$$

Then  $\Phi_h$  given by the formula (1) satisfies the Condition 6. (Recall  $l_4$  given in Lemma 4.15.)

**Proof.** Let us first consider  $1 - G_{m,\mathbb{B}}$ ,  $1 - G_{m,\mathbb{A}}$ . The first condition of *Condition 6* for  $1 - G_{m,\mathbb{B}}$ ,  $1 - G_{m,\mathbb{A}}$  is already checked. From Lemma 4.8 and Lemma D.1, the second property of *Condition 6* holds for  $1 - G_{m,\mathbb{A}}$ . As the corresponding property holds for  $1 - G_{m,\mathbb{B}}$  from Lemma 3.25 of [O1], the latter property holds also for  $1 - G_{m,\mathbb{B}}$ .

Let us consider  $1 - G_m$ . From Lemma 4.20 and *Condition 6* for  $1 - G_{m,\mathbb{A}}$ , we obtain *Condition 6* for  $1 - G_m$ .  $\square$

**Definition 4.23.** Let  $\Phi_0, \Phi_1 \in \mathcal{J}$  be positive interactions. For each  $\Lambda \in \mathfrak{S}_{\mathbb{Z}}$  and  $i = 0, 1$ , we denote by  $G_{\Lambda,i}$  the orthogonal projection onto  $(\ker H_{\Phi_i})_\Lambda$ . We say the pair  $(\Phi_0, \Phi_1)$  satisfies *Condition 7* if the followings hold.

1. The Hamiltonians  $H_{\Phi_0}, H_{\Phi_1}$  are frustration free.
2. The Hamiltonian  $H_{\Phi_1}$  is gapped with respect to the open boundary conditions.
3. There exist constants  $C > 0$  and  $0 < r < 1$  such that

$$\|G_{[0,N-1],0}G_{[0,N-1],1} - G_{[0,N-1],1}\| \leq Cr^N, \quad \text{for all } N \in \mathbb{N}. \quad (59)$$

4. There exists a state  $\omega_\infty$  on  $\mathcal{A}_{\mathbb{Z}}$  such that  $\mathcal{S}_{\mathbb{Z}}(H_{\Phi_0}) = \mathcal{S}_{\mathbb{Z}}(H_{\Phi_1}) = \{\omega_\infty\}$ .
5. For any  $M \in \mathbb{N}$  and  $\varphi \in \widetilde{\mathcal{S}_{\mathbb{Z},[-M,M]^c}(H_{\Phi_1})}$ ,  $\varphi$  is quasi-equivalent to  $\omega_\infty$ .

**Definition 4.24.** Let  $\Phi_0, \Phi_1 \in \mathcal{J}$  be positive interactions. We say the pair  $(\Phi_0, \Phi_1)$  satisfies *Condition 8* if it satisfies *Condition 7* and either (i) or (ii) of the following conditions holds.



- (i) (a) There exists a unique  $\alpha_{\Phi_0}$ -ground state, and  
 (b)  $\Phi_0 \in \mathcal{J}_B$ .
- (ii) (a) The Hamiltonian  $H_{\Phi_0}$  is gapped with respect to the open boundary conditions, and  
 (b) for any  $M \in \mathbb{N}$  and  $\varphi \in \widehat{\mathcal{S}_{\mathbb{Z}, [-M, M]^c}(H_{\Phi_0})}$ ,  $\varphi$  is quasi-equivalent to  $\omega_\infty$ . (Here,  $\omega_\infty$  is the state given by 4. of Condition 7.)

From Lemma 4.9, Lemma 4.13, Lemma 4.15, Lemma 4.16, Lemma 4.20 and Lemma 4.22, we have the following.

**Lemma 4.25.** *Let  $m_1, m_2$  with  $\max\{l_3, l_4\} \leq m_1$  and  $2l_{\mathbb{A}} \leq m_2$ . Then  $(\Phi_{1-G_{m_1}}, \Phi_{m_2, \mathbb{A}})$  satisfies Condition 8.*

By Corollary 1.4 of [O2], Theorem 1.18 of [O1], Lemma 4.9, Lemma 4.13, and Lemma 4.22, we obtain the following.

**Lemma 4.26.** *Let  $m_1, m_2$  with  $2l_{\mathbb{A}} \leq m_1$  and  $2l_{\mathbb{V}} \leq m_2$ . Then  $(\Phi_{m_1, \mathbb{A}}, \Phi_{m_2, \mathbb{V}})$  satisfies Condition 8.*

## 4.5 The overlaps of $G_{N, \mathbb{B}(t)}$

Throughout this subsection let  $2 \leq n_0 \in \mathbb{N}$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , and  $\mathbb{A}, \mathbb{B}(t)$  be given by (23), (22). As  $\mathbb{B}(t)$  belongs to Class A if  $t \in (0, 1]$  from Lemma 4.1, we obtain  $e_{\mathbb{B}(t)}$  and  $\varphi_{\mathbb{B}(t)}$  given in Proposition 3.1 of [O1]. In this subsection, we show the following Lemma.

**Lemma 4.27.** *There exist  $C_9$ ,  $0 < s_4 < 1$ , and  $l_5 \in \mathbb{N}$  such that*

$$\sup_{t \in [0, 1]} \left\| \left( \mathbb{I}_{[0, N-l]} \otimes \hat{G}_l(t) \right) \left( \hat{G}_N(t) \otimes \mathbb{I}_{\{N\}} - \hat{G}_{N+1}(t) \right) \right\| \leq n_0^2 (k_R + 1) (k_L + 1) C_9 (s_4^l + s_4^{N-l}), \quad (60)$$

for all  $N, l \in \mathbb{N}$  such that  $N \geq l \geq \max\{l_3, l_5, n_0^6 (k_R + 1) (k_L + 1)\}$ . (Recall  $l_3$  given in Lemma 4.13.)

First we estimate various sesqui-linear forms.

**Lemma 4.28.** *There exist constants  $C_3, C_4 > 0$ ,  $0 < s_2 < 1$  such that*

$$\left| \left\langle \Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l, t}(X_1), \Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l, t}(X_2) \right\rangle - \varphi_{\mathbb{B}(t)}(X_1^* a_t e_{\mathbb{B}(t)} a_t X_2) \right| \leq C_3 s_2^l \|X_1\| \|X_2\|, \quad (61)$$

$$\left| \left\langle \Gamma_{l, \mathbb{B}(t)}^{(R)}(\Theta_{l+1, t}(X_1) B_\mu(t)^*), \Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l, t}(X_2) \right\rangle - \varphi_{\mathbb{B}(t)}(B_\mu(t) X_1^* a_t e_{\mathbb{B}(t)} a_t X_2) \right| \leq C_3 s_2^l \|X_1\| \|X_2\|, \quad (62)$$

$$|\varphi_{\mathbb{B}(t)}(X_1^* a_t e_{\mathbb{B}(t)} a_t X_2)| \leq C_4 \|X_1\| \|X_2\|, \quad (63)$$

$$|\varphi_{\mathbb{B}(t)}(B_\mu(t) X_1^* a_t e_{\mathbb{B}(t)} a_t X_2)| \leq C_4 \|X_1\| \|X_2\|, \quad (64)$$

for all  $t \in (0, 1]$ ,  $l \in \mathbb{N}$ ,  $X_1, X_2 \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R + k_L + 1} P_L^{(k_R, k_L)}$ ,  $\mu \in \{1, \dots, n\}$ .

**Proof.** We prove (62) and (64). The proofs of (61) and (63) are the same. First, from the definition, we have

$$\Gamma_{l, \mathbb{B}(t)}^{(R)}(\Theta_{l+1, t}(X_1) B_\mu(t)^*) = \sum_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \left( \text{Tr } \Theta_{l+1, t}(X_1) (B_{\mu^{(l)}}(t) B_\mu(t))^* \right) \widehat{\psi_{\mu^{(l)}}}.$$

In the Tr on the right hand side, we have  $l+1$  number of  $B_\mu(t)$ s and  $\Theta_{l+1,t}(X_1)$ . Therefore, from the proof of Lemma 4.4, we see that it is of the form  $r_{T_{\tilde{\omega}(t)}}^{-\frac{l+1}{2}} \times$  a polynomial of  $t$ . Similarly,  $\Gamma_{l,\mathbb{B}(t)}^{(R)} \circ \Theta_{l,t}(X_2)$  is of the form  $r_{T_{\tilde{\omega}(t)}}^{-\frac{l}{2}} \times$  a polynomial of  $t$ . Therefore, we have

$$\lim_{t \downarrow 0} t \left\langle \Gamma_{l,\mathbb{B}(t)}^{(R)} (\Theta_{l+1,t}(X_1) B_\mu(t)^*), \Gamma_{l,\mathbb{B}(t)}^{(R)} \circ \Theta_{l,t}(X_2) \right\rangle = 0. \quad (65)$$

Next, let  $0 < s_1 < 1$  be the constant given in Lemma 4.3. and set  $S_1 := \{z \in \mathbb{C} \mid |z| = \frac{1+s_1}{2}\} \cup \{z \in \mathbb{C} \mid |z-1| = \frac{1-s_1}{2}\}$ . Then by Spectral Property II of  $T_{\mathbb{B}(t)}$ ,  $t \in [0, 1]$ , and the definition of  $s_1$  from Lemma 4.3, we have

$$T_{\mathbb{B}(t)}^l = P_{\{1\}}^{T_{\mathbb{B}(t)}} + T_{\mathbb{B}(t)}^l \left( \text{id} - P_{\{1\}}^{T_{\mathbb{B}(t)}} \right) = \left( \oint_{|z-1|=\frac{1-s_1}{2}} dz + \oint_{|z|=\frac{1+s_1}{2}} dz z^l \right) (z - T_{\mathbb{B}(t)})^{-1} \quad (66)$$

From this and a routine calculation from [O1], we obtain

$$\begin{aligned} & \left\langle \Gamma_{l,\mathbb{B}(t)}^{(R)} (\Theta_{l+1,t}(X_1) B_\mu(t)^*), \Gamma_{l,\mathbb{B}(t)}^{(R)} \circ \Theta_{l,t}(X_2) \right\rangle \\ &= \sum_{\alpha, \beta=1}^{n_0} \sum_{i, j=-k_R}^0 \left\langle \chi_\alpha^{(n_0)} \otimes f_i^{(k_R, k_L)}, T_{\mathbb{B}(t)}^l \left( B_\mu(t) \Theta_{l+1,t}(X_1)^* \left| \chi_\alpha^{(n_0)} \otimes f_i^{(k_R, k_L)} \right\rangle \left\langle \chi_\beta^{(n_0)} \otimes f_j^{(k_R, k_L)} \right| \Theta_{l,t}(X_2) \right) \chi_\beta^{(n_0)} \otimes f_j^{(k_R, k_L)} \right\rangle \\ &= \left( \oint_{|z-1|=\frac{1-s_1}{2}} dz + \oint_{|z|=\frac{1+s_1}{2}} dz z^l \right) \sum_{\alpha, \beta=1}^{n_0} \sum_{i, j=-k_R}^0 \\ & \left\langle \chi_\alpha^{(n_0)} \otimes f_i^{(k_R, k_L)}, (z - T_{\mathbb{B}(t)})^{-1} \left( B_\mu(t) \Theta_{l+1,t}(X_1)^* \left| \chi_\alpha^{(n_0)} \otimes f_i^{(k_R, k_L)} \right\rangle \left\langle \chi_\beta^{(n_0)} \otimes f_j^{(k_R, k_L)} \right| \Theta_{l,t}(X_2) \right) \chi_\beta^{(n_0)} \otimes f_j^{(k_R, k_L)} \right\rangle \end{aligned} \quad (67)$$

for each  $t \in (0, 1]$ . Now, we expand  $(z - T_{\mathbb{B}(t)})^{-1}$ ,  $B_\mu(t)$ , as  $(z - T_{\mathbb{B}(t)})^{-1} = \sum_{j=0,1} t^j A_j(z) + t^2 A_2(z, t)$ ,  $B_\mu(t) = \sum_{j=0,1} t^j W_j + t^2 W_2(t)$ ,  $t \in [0, 1]$ ,  $z \in S_1$ . Here  $A_j(z)$ ,  $A_2(z, t)$ ,  $W_j$ ,  $W_2(t)$  are operators on  $M_{n_0} \otimes M_{k_R+k_L+1}$  such that  $\sup_{z \in S_1} \|A_j(z)\|$ ,  $\sup_{t \in [0,1], z \in S_1} \|A_2(z, t)\|$ ,  $\sup_{t \in [0,1]} \|W_2(t)\|$  are finite. Substituting these in (67), the right hand side of (67) can be expanded as  $t^{-2} b_2(l, X_1, X_2) + t^{-1} b_1(l, X_1, X_2) + b_0(t, l, X_1, X_2)$ . Here, for the map  $b_0 : (0, 1] \times \mathbb{N} \times (M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)})^{\times 2} \ni (t, l, X_1, X_2) \mapsto b_0(t, l, X_1, X_2) \in \mathbb{C}$ , there exists a constant  $C_4 > 0$  such that

$$\sup_{t \in (0,1]} \sup_{l \in \mathbb{N}} |b_0(t, l, X_1, X_2)| \leq C_4 \|X_1\| \|X_2\|. \quad (68)$$

Furthermore, there exists a limit  $b_\infty(t, X_1, X_2) = \lim_{l \rightarrow \infty} b_0(t, l, X_1, X_2)$ . From (68), we have

$$\sup_{t \in (0,1]} |b_\infty(t, X_1, X_2)| \leq C_4 \|X_1\| \|X_2\|. \quad (69)$$

There exist constants  $C_3 > 0$  and  $0 < s_2 < 1$  such that

$$\sup_{t \in (0,1]} |b_\infty(t, X_1, X_2) - b_0(t, l, X_1, X_2)| \leq C_3 s_2^l \|X_1\| \|X_2\|, \quad X_1, X_2 \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)} \quad l \in \mathbb{N}. \quad (70)$$

On the other hand, from the property (65), we know that  $\lim_{t \downarrow 0} t^{-1} b_2(l, X_1, X_2) + b_1(l, X_1, X_2) + t b_0(t, l, X_1, X_2) = 0$ . This means  $b_2(l, X_1, X_2) = b_1(l, X_1, X_2) = 0$ . Substituting these, we obtain

$$\left\langle \Gamma_{l,\mathbb{B}(t)}^{(R)} (\Theta_{l+1,t}(X_1) B_\mu(t)^*), \Gamma_{l,\mathbb{B}(t)}^{(R)} \circ \Theta_{l,t}(X_2) \right\rangle = b_0(t, l, X_1, X_2). \quad (71)$$

Take  $l \rightarrow \infty$  limit of this equation. From (70) and the first equality of (67) combined with the spectral property of  $T_{\mathbb{B}(t)}$ , we obtain

$$\varphi_{\mathbb{B}(t)}(B_\mu(t)X_1^*a_te_{\mathbb{B}(t)}a_tX_2) = b_\infty(t, X_1, X_2). \quad (72)$$

From (70), (71), and (72), we obtain (62). Furthermore, from (69) and (72), we obtain (64).  $\square$

**Lemma 4.29.** *There exist constants  $C_5, C_6 > 0$  such that*

$$C_5 \operatorname{Tr} X^*X \leq \varphi_{\mathbb{B}(t)}(X^*a_te_{\mathbb{B}(t)}a_tX) \leq C_6 \operatorname{Tr} X^*X, \quad X \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}. \quad (73)$$

**Proof.** We define a linear map  $\iota : M_{n_0} \rightarrow M_{2n_0-1}$  by  $\iota(e_{\alpha\beta}^{(n_0)}) = E_{-(\alpha-1), \beta-1}^{(n_0-1, n_0-1)}$ ,  $\alpha, \beta = 1, \dots, n_0$ . By Lemma 4.12, we have

$$\left\| \Gamma_{l, \mathbb{A}}^{(R)} \circ (\iota \otimes \operatorname{id})(X) - \Gamma_{l, \mathbb{A}'}^{(R)} \circ (\Theta_l \otimes \operatorname{id})(X) \right\| \leq n_0 C_7 \kappa^{\frac{l}{2}} \|X\|, \quad l \in \mathbb{N}, \quad X \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}. \quad (74)$$

By (33) of [O1] with Lemma 4.6, and (61), (74), we have a ( $t$ -independent) sequence of positive numbers  $\delta_l$ ,  $l \in \mathbb{N}$  with  $\delta_l \rightarrow 0$ ,  $l \rightarrow \infty$ , such that

$$\begin{aligned} & \left| \varphi_{\mathbb{B}(t)}(X_0^*a_te_{\mathbb{B}(t)}a_tX_1) - \varphi_{\mathbb{A}}(((\iota \otimes \operatorname{id})(X_0))^* e_{\mathbb{A}} (\iota \otimes \operatorname{id})(X_1)) \right| \\ & \leq \left| \left\langle \Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l,t}(X_0), \Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l,t}(X_1) \right\rangle - \left\langle \Gamma_{l, \mathbb{A}'}^{(R)} \circ (\Theta_l \otimes \operatorname{id})(X_0), \Gamma_{l, \mathbb{A}'}^{(R)} \circ (\Theta_l \otimes \operatorname{id})(X_1) \right\rangle \right| + \delta_l. \end{aligned}$$

We first take  $\limsup_{t \downarrow 0}$ , applying (55) for this inequality, and then take  $l \rightarrow \infty$  limit. Hence we obtain

$$\lim_{t \downarrow 0} \varphi_{\mathbb{B}(t)}(X_0^*a_te_{\mathbb{B}(t)}a_tX_1) = \varphi_{\mathbb{A}}(((\iota \otimes \operatorname{id})(X_0))^* e_{\mathbb{A}} (\iota \otimes \operatorname{id})(X_1)), \quad X_0, X_1 \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}. \quad (75)$$

As  $\iota \otimes \operatorname{id}$  is a bijection from  $M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$  to  $P_{R, \mathbb{A}}(M_{2n_0-1} \otimes M_{k_R+k_L+1}) P_{L, \mathbb{A}}$  and  $s(e_{\mathbb{A}}) = P_{R, \mathbb{A}}$  and  $s(\varphi_{\mathbb{A}}) = P_{L, \mathbb{A}}$ ,  $(X_0, X_1) \mapsto \varphi_{\mathbb{A}}(((\iota \otimes \operatorname{id})(X_0))^* e_{\mathbb{A}} (\iota \otimes \operatorname{id})(X_1))$ ,  $X_0, X_1 \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$  defines an inner product on  $M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ . From this and (75), and the finite dimensionality of  $M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ , we prove the claim of the Lemma.  $\square$

From the upper bound of the last Lemma, we obtain the following.

**Corollary 4.30.** *We have  $\sup_{t \in (0,1]} \|a_te_{\mathbb{B}(t)}a_t\| < \infty$ .*

Furthermore, from Lemma 4.28 and Lemma 4.29, we obtain the following.

**Corollary 4.31.** *Let  $C_5$  be the constant given in Lemma 4.29. Then there exists an  $l_5 \in \mathbb{N}$  such that*

$$\frac{C_5}{2} \operatorname{Tr} X^*X \leq \left\| \Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l,t}(X) \right\|^2, \quad X \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}, \quad l_5 \leq l, \quad t \in (0,1]. \quad (76)$$

For any  $N, l \in \mathbb{N}$  with  $N \geq l \geq 2$  and  $t \in (0, 1]$ ,  $Z, X_\mu \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ ,  $\mu = 1, \dots, n$ , we define

$$\mathcal{Q}(t, N, l, Z, \{X_\mu\}_\mu) := \sum_{\mu^{(N-l+1)} \in \{1, \dots, n\} \times N-l+1} \left| \sum_{\mu=1}^n \left\langle \Gamma_{l-1, \mathbb{B}(t)}^{(R)} (\Theta_{l,t}(Z) B_\mu(t)^*), \Gamma_{l-1, \mathbb{B}(t)}^{(R)} \left( \widehat{(B_{\mu^{(N-l+1)}}(t))^*} \Theta_{N,t}(X_\mu) \right) \right\rangle \right|^2,$$

$$\mathcal{R}(t, Z, \{X_\mu\}_\mu) := \sum_{\mu=1}^n a_t X_\mu \rho_{\mathbb{B}(t)} B_\mu(t) Z^* a_t e_{\mathbb{B}(t)} \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_R^{(k_R, k_L)}.$$

**Lemma 4.32.** *There exist constants  $C_{10} > 0$ ,  $0 < s_3 < 1$  satisfying the followings.: For any  $N, l \in \mathbb{N}$  with  $N \geq l \geq 2$ ,  $t \in (0, 1]$ , and  $Z, X_\mu \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ ,  $\mu = 1, \dots, n$ , we have*

$$|\mathcal{Q}(t, N, l, Z, \{X_\mu\}_\mu) - \varphi_{\mathbb{B}(t)}((\mathcal{R}(t, Z, \{X_\mu\}_\mu))^* e_{\mathbb{B}(t)} \mathcal{R}(t, Z, \{X_\mu\}_\mu))| \leq C_{10} (s_3^l + s_3^{N-l}) \left( \sum_{\mu=1}^n \|X_\mu\|^2 \right) \|Z\|^2. \quad (77)$$

**Proof.** The proof is analogous to that of Lemma 4.28. Let  $0 < s_1 < 1$  be the constant given in Lemma 4.3. In this proof, we use a notation

$$\int_{N,l} := \frac{1}{(2\pi i)^3} \left( \oint_{|z|=\frac{1+s_1}{2}} dz z^{N-l+1} + \oint_{|z-1|=\frac{1-s_1}{2}} dz \right) \left( \oint_{|\zeta|=\frac{1+s_1}{2}} d\zeta \zeta^{l-1} + \oint_{|\zeta-1|=\frac{1-s_1}{2}} d\zeta \right) \left( \oint_{|w|=\frac{1+s_1}{2}} dw w^{l-1} + \oint_{|w-1|=\frac{1-s_1}{2}} dw \right),$$

Note that  $\Gamma_{l-1, \mathbb{B}(t)}^{(R)} (\Theta_{l,t}(Z) B_\mu(t)^*)$  and  $\Gamma_{l-1, \mathbb{B}(t)}^{(R)} \left( \widehat{(B_{\mu^{(N-l+1)}}(t))^*} \Theta_{N,t}(X_\mu) \right)$  are of the form  $r_{T_{\tilde{\omega}(t)}}^{-\frac{l}{2}} \times$  a polynomial of  $t$  and  $r_{T_{\tilde{\omega}(t)}}^{-\frac{N}{2}} \times$  a polynomial of  $t$ , from the proof of Lemma 4.4. Therefore, we have

$$\lim_{t \downarrow 0} t \cdot \mathcal{Q}(t, N, l, Z, \{X_\mu\}_\mu) = 0. \quad (78)$$

From (66) and a routine calculation from [O1], we obtain

$$\begin{aligned} & \mathcal{Q}(t, N, l, Z, \{X_\mu\}_\mu) \\ &= \sum_{\mu, \mu'=1}^n \sum_{\alpha, \beta=1}^{n_0} \sum_{i, j=-k_R}^0 \sum_{\alpha', \beta'=1}^{n_0} \sum_{i', j'=-k_R}^0 \\ & \left\langle \chi_{\beta'}^{(n_0)} \otimes f_{j'}^{(k_R, k_L)}, T_{\mathbb{B}(t)}^{N-l+1} \left( \begin{aligned} & T_{\mathbb{B}(t)}^{l-1} \left( \Theta_{N,t}(X_{\mu'})^* \left| \chi_{\beta'}^{(n_0)} \otimes f_{j'}^{(k_R, k_L)} \right\rangle \left\langle \chi_{\alpha'}^{(n_0)} \otimes f_{i'}^{(k_R, k_L)} \right| \Theta_{lt}(Z) B_{\mu'}(t)^* \right) \\ & \cdot \left| \chi_{\alpha'}^{(n_0)} \otimes f_{i'}^{(k_R, k_L)} \right\rangle \left\langle \chi_{\alpha}^{(n_0)} \otimes f_i^{(k_R, k_L)} \right| \\ & \cdot T_{\mathbb{B}(t)}^{l-1} \left( B_\mu(t) \Theta_{lt}(Z)^* \left| \chi_{\alpha}^{(n_0)} \otimes f_i^{(k_R, k_L)} \right\rangle \left\langle \chi_{\beta}^{(n_0)} \otimes f_j^{(k_R, k_L)} \right| \Theta_{N,t}(X_\mu) \right) \end{aligned} \right) \left( \chi_{\beta}^{(n_0)} \otimes f_j^{(k_R, k_L)} \right) \right\rangle \end{aligned} \quad (79)$$

$$\begin{aligned} &= \widehat{\int_{N,l}} \sum_{\mu, \mu'=1}^n \sum_{\alpha, \beta=1}^{n_0} \sum_{i, j=-k_R}^0 \sum_{\alpha', \beta'=1}^{n_0} \sum_{i', j'=-k_R}^0 \\ & \left\langle \chi_{\beta'}^{(n_0)} \otimes f_{j'}^{(k_R, k_L)}, \right. \\ & (z - T_{\mathbb{B}(t)})^{-1} \left( \begin{aligned} & (\zeta - T_{\mathbb{B}(t)})^{-1} \left( \Theta_{N,t}(X_{\mu'})^* \left| \chi_{\beta'}^{(n_0)} \otimes f_{j'}^{(k_R, k_L)} \right\rangle \left\langle \chi_{\alpha'}^{(n_0)} \otimes f_{i'}^{(k_R, k_L)} \right| \Theta_{lt}(Z) B_{\mu'}(t)^* \right) \\ & \cdot \left| \chi_{\alpha'}^{(n_0)} \otimes f_{i'}^{(k_R, k_L)} \right\rangle \left\langle \chi_{\alpha}^{(n_0)} \otimes f_i^{(k_R, k_L)} \right| \\ & \cdot (w - T_{\mathbb{B}(t)})^{-1} \left( B_\mu(t) \Theta_{lt}(Z)^* \left| \chi_{\alpha}^{(n_0)} \otimes f_i^{(k_R, k_L)} \right\rangle \left\langle \chi_{\beta}^{(n_0)} \otimes f_j^{(k_R, k_L)} \right| \Theta_{N,t}(X_\mu) \right) \end{aligned} \right) \left( \chi_{\beta}^{(n_0)} \otimes f_j^{(k_R, k_L)} \right) \right\rangle \end{aligned} \quad (80)$$

for each  $t \in (0, 1]$ . Now, as in the proof of Lemma 4.28, expanding  $(z - T_{\mathbb{B}(t)})^{-1}$ ,  $(w - T_{\mathbb{B}(t)})^{-1}$ ,  $(\zeta - T_{\mathbb{B}(t)})^{-1}$ ,  $B_\mu(t)$  we obtain an expansion

$$\mathcal{Q}(t, N, l, Z, \{X_\mu\}_\mu) = \sum_{j=1}^4 t^{-j} b_j(N, l, Z, \{X_\mu\}_\mu) + b_0(t, N, l, Z, \{X_\mu\}_\mu). \quad (81)$$

By taking the limit  $N \rightarrow \infty$  and then  $l \rightarrow \infty$  after that for  $b_0(t, N, l, Z, \{X_\mu\})$  we obtain a function  $b_\infty(t, Z, \{X_\mu\}) = \lim_{l \rightarrow \infty} (\lim_{N \rightarrow \infty} b_0(t, N, l, Z, \{X_\mu\}))$ . There exist constants  $C_{10} > 0$  and  $0 < s_3 < 1$  such that

$$\sup_{t \in (0, 1]} \sup_{N, l \in \mathbb{N}; N \geq l} |b_0(t, N, l, Z, \{X_\mu\})|, \max_{j=1, \dots, 4} \sup_{N, l \in \mathbb{N}; N \geq l} |b_j(N, l, Z, \{X_\mu\})|, \sup_{t \in (0, 1]} |b_\infty(t, Z, \{X_\mu\})| \leq C_{10} \sum_{\mu=1}^n \|X_\mu\|^2 \cdot \|Z\|^2, \quad (82)$$

$$\sup_{t \in (0, 1]} \sup_{N, l \in \mathbb{N}; N \geq l} |b_\infty(t, Z, \{X_\mu\}) - b_0(t, N, l, Z, \{X_\mu\})| \leq C_{10} (s_3^l + s_3^{N-l}) \sum_{\mu=1}^n \|X_\mu\|^2 \cdot \|Z\|^2, \quad (83)$$

for any  $Z, X_\mu \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ .

The expansion (81) and the property (78) implies  $b_j(N, l, Z, \{X_\mu\}) = 0$  for any  $j = 1, \dots, 4$ ,  $N, l \in \mathbb{N}$  with  $N \geq l$ ,  $t \in (0, 1]$  and  $Z, X_\mu \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ . Hence we obtain

$$\mathcal{Q}(t, N, l, Z, \{X_\mu\}_\mu) = b_0(t, N, l, Z, \{X_\mu\}), \quad (84)$$

for any  $N, l \in \mathbb{N}$  with  $N \geq l$ ,  $t \in (0, 1]$ ,  $Z, X_\mu \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ ,  $\mu = 1, \dots, n$ . Take  $N \rightarrow \infty$  limit and  $l \rightarrow \infty$  then after in this equation. From (79) combined with the spectral property of  $T_{\mathbb{B}(t)}$ , we obtain

$$\varphi_{\mathbb{B}(t)}((\mathcal{R}(t, Z, \{X_\mu\}_\mu))^* e_{\mathbb{B}(t)} \mathcal{R}(t, Z, \{X_\mu\}_\mu)) = b_\infty(t, Z, \{X_\mu\}). \quad (85)$$

From (83), (84), and (85), we obtain (77).  $\square$

**Lemma 4.33.** *There exist constants  $C_9 > 0$ ,  $0 < s_4 < 1$  satisfying the followings.: For any  $N, l \in \mathbb{N}$  with  $N \geq l \geq \max\{l_5, n_0^6(k_R + 1)(k_L + 1)\}$ ,  $t \in (0, 1]$ ,  $Z \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ ,  $\Phi \in (\mathcal{G}_{N, \mathbb{B}(t)} \otimes \mathbb{C}^n) \cap \mathcal{G}_{N+1, \mathbb{B}(t)}^\perp$  we have*

$$\left\| \left( \mathbb{I}_{N+1-l} \otimes \left\langle \Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l, t}(Z) \right\rangle \right) \Phi \right\|^2 \leq C_9 (s_4^l + s_4^{N-l}) \|\Phi\|^2 \left\| \Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l, t}(Z) \right\|^2. \quad (86)$$

(Recall  $l_5$  given in Corollary 4.31.)

**Proof.** Let  $N, l \in \mathbb{N}$  with  $N \geq l \geq \max\{l_5, n_0^6(k_R + 1)(k_L + 1)\}$ ,  $t \in (0, 1]$ ,  $Z \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ , and  $\Phi \in (\mathcal{G}_{N, \mathbb{B}(t)} \otimes \mathbb{C}^n) \cap \mathcal{G}_{N+1, \mathbb{B}(t)}^\perp$ . As  $\mathbb{B}(t) \in \text{Class}(n, n_0, k_R, k_L)$ , and  $N \geq n_0^6(k_R + 1)(k_L + 1) \geq l_{\mathbb{B}(t)}$ , from Proposition 3.1 of [O1], for  $\Phi \in \mathcal{G}_{N, \mathbb{B}(t)} \otimes \mathbb{C}^n$  there exist  $\{\Phi_\mu\}_{\mu=1}^n \subset M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$  such that  $\Phi = \sum_{\mu=1}^n \Gamma_{N, \mathbb{B}(t)}^{(R)} \circ \Theta_{Nt}(\Phi_\mu) \otimes \psi_\mu$ . By Corollary 4.31, we have

$$\sum_{\mu=1}^n \|\Phi_\mu\|^2 \leq \frac{2}{C_5} \sum_{\mu=1}^n \left\| \Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l, t}(\Phi_\mu) \right\|^2 = \frac{2}{C_5} \|\Phi\|^2. \quad (87)$$

We claim

$$\left| \sum_{\mu=1}^n \varphi_{\mathbb{B}(t)}(B_\mu(t) X^* a_t e_{\mathbb{B}(t)} a_t \Phi_\mu) \right| \leq C_3 \cdot \sqrt{\frac{2n}{C_5}} \cdot s_1^N \cdot \|\Phi\| \|X\|, \quad X \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}. \quad (88)$$

To see this, let  $X \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ . From  $\Phi \in \mathcal{G}_{N+1, \mathbb{B}(t)}^\perp$ , we have

$$0 = \left\langle \Gamma_{N+1, \mathbb{B}(t)}^{(R)} \circ \Theta_{N+1, t}(X), \Phi \right\rangle = \sum_{\mu=1}^n \left\langle \Gamma_{N, \mathbb{B}(t)}^{(R)} (\Theta_{N+1, t}(X) B_\mu(t)^*), \Gamma_{N, \mathbb{B}(t)}^{(R)} \circ \Theta_{Nt}(\Phi_\mu) \right\rangle.$$

From this, we obtain

$$\begin{aligned} & \left| \sum_{\mu=1}^n \varphi_{\mathbb{B}(t)}(B_\mu(t) X^* a_t e_{\mathbb{B}(t)} a_t \Phi_\mu) \right| \\ &= \left| \sum_{\mu=1}^n \left( \varphi_{\mathbb{B}(t)}(B_\mu(t) X^* a_t e_{\mathbb{B}(t)} a_t \Phi_\mu) - \left\langle \Gamma_{N, \mathbb{B}(t)}^{(R)} (\Theta_{N+1, t}(X) B_\mu(t)^*), \Gamma_{N, \mathbb{B}(t)}^{(R)} \circ \Theta_{Nt}(\Phi_\mu) \right\rangle \right) \right| \leq C_3 s_2^N \|X\| \sum_{\mu=1}^n \|\Phi_\mu\| \\ &\leq C_3 s_2^N \|X\| \sqrt{\frac{2n}{C_5}} \|\Phi\|. \end{aligned}$$

In the first inequality, we used Lemma 4.28. The second inequality is due to (87) and Cauchy-Schwartz inequality. This proves the claim (88).

From Corollary 4.30,

$$C' := \sup_{t \in (0, 1]} \left( \|e_{\mathbb{B}(t)}\| \|a_t e_{\mathbb{B}(t)} a_t\| \left( \sum_{\mu=1}^n \|B_\mu(t)\|^2 \right)^{\frac{1}{2}} \right), \quad C'' := \frac{2\sqrt{n}}{C_5} C_3 C'$$

are finite positive constants. We apply (88) to

$$S_t := \sum_{\mu'=1}^n \Phi_{\mu'} \rho_{\mathbb{B}(t)} B_{\mu'}(t) Z^* a_t e_{\mathbb{B}(t)} \rho_{\mathbb{B}(t)} e_{\mathbb{B}(t)} a_t Z \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}.$$

Note that, by  $\text{Tr } \rho_{\mathbb{B}(t)} = 1$  and Cauchy-Schwartz inequality and (87), we have

$$\|S_t\| \leq C' \cdot \|Z\|^2 \left( \sum_{\mu=1}^n \|\Phi_\mu\|^2 \right)^{\frac{1}{2}} \leq C' \cdot \|Z\|^2 \sqrt{\frac{2}{C_5}} \|\Phi\|.$$

Applying (88) to  $S_t$ , we obtain

$$\begin{aligned} & \left| \varphi_{\mathbb{B}(t)}((\mathcal{R}(t, Z, \{\Phi_\mu\}_\mu))^* e_{\mathbb{B}(t)} \mathcal{R}(t, Z, \{\Phi_\mu\}_\mu)) \right| = \left| \sum_{\mu=1}^n \varphi_{\mathbb{B}(t)}(B_\mu(t) S_t^* a_t e_{\mathbb{B}(t)} a_t \Phi_\mu) \right| \\ &\leq C_3 \cdot \sqrt{\frac{2n}{C_5}} \cdot s_1^N \cdot \|\Phi\| \|S_t\| \leq C'' s_1^N \|Z\|^2 \|\Phi\|^2. \end{aligned} \tag{89}$$

Now we are ready to show (86). It is straightforward to check that  $\left\| \left( \mathbb{I}_{N+1-l} \otimes \left\langle \Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l, t}(Z) \right\rangle \right) \Phi \right\|^2$  is equal to  $\mathcal{Q}(t, N, l, Z, \{\Phi_\mu\}_\mu)$ . From this, applying Lemma 4.32 and (89),

$$\begin{aligned} & \left\| \left( \mathbb{I}_{N+1-l} \otimes \left\langle \Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l, t}(Z) \right\rangle \right) \Phi \right\|^2 = \mathcal{Q}(t, N, l, Z, \{\Phi_\mu\}_\mu) \\ &\leq \left| \mathcal{Q}(t, N, l, Z, \{\Phi_\mu\}_\mu) - \varphi_{\mathbb{B}(t)}((\mathcal{R}(t, Z, \{\Phi_\mu\}_\mu))^* e_{\mathbb{B}(t)} \mathcal{R}(t, Z, \{\Phi_\mu\}_\mu)) \right| + \varphi_{\mathbb{B}(t)}((\mathcal{R}(t, Z, \{\Phi_\mu\}_\mu))^* e_{\mathbb{B}(t)} \mathcal{R}(t, Z, \{\Phi_\mu\}_\mu)) \\ &\leq C_{10} (s_3^l + s_3^{N-l}) \left( \sum_{\mu=1}^n \|\Phi_\mu\|^2 \right) \|Z\|^2 + C'' s_1^N \|Z\|^2 \|\Phi\|^2 \leq C_{10} (s_3^l + s_3^{N-l}) \frac{2}{C_5} \|\Phi\|^2 \|Z\|^2 + C'' s_2^N \|Z\|^2 \|\Phi\|^2. \end{aligned}$$

In the last inequality, we used (87). This and Corollary 4.31 proves the claim of Lemma.  $\square$

**Proof of Lemma 4.27.** Let  $C_9$ ,  $0 < s_4 < 1$  be the constants in Lemma 4.33, and  $l_5 \in \mathbb{N}$  in Corollary 4.31. Let  $N, l \in \mathbb{N}$  such that  $N \geq l \geq \max\{l_5, n_0^6(k_R + 1)(k_L + 1)\}$ .

For each  $t \in (0, 1]$ , we have  $\mathbb{B}(t) \in \text{Class}(n, n_0, k_R, k_L)$  and  $l \geq n_0^6(k_R + 1)(k_L + 1) \geq l_{\mathbb{B}(t)}$ . Therefore, by Proposition 3.1 of [O1], there exist  $Z_{t,i} \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R + k_L + 1} P_L^{(k_R, k_L)}$ ,  $i = 1, \dots, n_0^2(k_R + 1)(k_L + 1)$  such that  $\{\Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l,t}(Z_{t,i})\}_{i=1}^{n_0^2(k_R + 1)(k_L + 1)}$  is a CONS of  $\mathcal{G}_{l, \mathbb{B}(t)}$ .

Let  $\xi \in \bigotimes_{i=0}^N \mathbb{C}^n$  and  $t \in (0, 1]$ . Then  $\Phi_t := (G_{N, \mathbb{B}(t)} \otimes \mathbb{I}_{\{N\}} - G_{N+1, \mathbb{B}(t)}) \xi \in (\mathcal{G}_{N, \mathbb{B}(t)} \otimes \mathbb{C}^n) \cap \mathcal{G}_{N+1, \mathbb{B}(t)}^\perp$ . From Lemma 4.33, we have

$$\begin{aligned} \left\| (\mathbb{I}_{[0, N-l]} \otimes G_{l, \mathbb{B}(t)}) (G_{N, \mathbb{B}(t)} \otimes \mathbb{I}_{\{N\}} - G_{N+1, \mathbb{B}(t)}) \xi \right\|^2 &= \sum_{i=1}^{n_0^2(k_R + 1)(k_L + 1)} \left\| \left( \mathbb{I}_{[0, N-l]} \otimes \left\langle \Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l,t}(Z_{t,i}) \right\rangle \right) \Phi_t \right\|^2 \\ &\leq C_9 (s_4^l + s_4^{N-l}) \sum_{i=1}^{n_0^2(k_R + 1)(k_L + 1)} \|\Phi_t\|^2 \left\| \Gamma_{l, \mathbb{B}(t)}^{(R)} \circ \Theta_{l,t}(Z_{t,i}) \right\|^2 \leq n_0^2(k_R + 1)(k_L + 1) C_9 (s_4^l + s_4^{N-l}) \|\xi\|^2. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \sup_{t \in [0, 1]} \left\| \left( \mathbb{I}_{[0, N-l]} \otimes \hat{G}_l(t) \right) \left( \hat{G}_N(t) \otimes \mathbb{I}_{\{N\}} - \hat{G}_{N+1}(t) \right) \right\| &= \sup_{t \in (0, 1]} \left\| (\mathbb{I}_{[0, N-l]} \otimes G_{l, \mathbb{B}(t)}) (G_{N, \mathbb{B}(t)} \otimes \mathbb{I}_{\{N\}} - G_{N+1, \mathbb{B}(t)}) \right\| \\ &\leq n_0^2(k_R + 1)(k_L + 1) C_9 (s_4^l + s_4^{N-l}). \end{aligned}$$

For the first equality, we used the continuity of  $\hat{G}_l(t)$ .  $\square$

**Lemma 4.34.** We define  $l_6 \in \mathbb{N}$  by

$$l_6 := \inf\{l' \in \mathbb{N} \mid l' \geq l_5, n_0^2(k_R + 1)(k_L + 1) C_9 s_4^l < \frac{1}{8\sqrt{l}}, \text{ for all } l \geq l'\}.$$

The paths  $\{\hat{\xi}_{\alpha, \beta, a, b}^{(l)}\}$  given in Lemma 4.14 satisfies the Condition 5 with respect to  $(\max\{l_3, n_0^6(k_R + 1)(k_L + 1)\}, \max\{l_4, 2n_0^6(k_R + 1)(k_L + 1)\}, l_6)$ . (Recall that  $C_9, s_4$  are given in Lemma 4.27, and  $l_3, l_4, l_5$  by Lemma 4.13, 4.15, 4.27, respectively.)

**Proof.** (i) of Condition 5 follows from Lemma 4.14. (ii) of Condition 5 for  $l' = \max\{l_3, n_0^6(k_R + 1)(k_L + 1)\}$  follows from Lemma 4.13 and Proposition 3.1 of [O1] with  $l_{\mathbb{B}(t)} \leq n_0^6(k_R + 1)(k_L + 1)$ . (iii) of Condition 5 for  $m' = \max\{l_4, 2n_0^6(k_R + 1)(k_L + 1)\}$  follows from Lemma 4.15 and Proposition 3.1 of [O1] with  $l_{\mathbb{B}(t)} \leq n_0^6(k_R + 1)(k_L + 1)$ . (iv) of Condition 5 with  $\varepsilon_l := 4n_0^2(k_R + 1)(k_L + 1) C_9 s_4^l$  and  $l'' = l_6$  is from Lemma 4.27.  $\square$

## 4.6 Proof of Theorem 1.8

In this subsection, we complete the proof of Theorem 1.8. We use the following notations.

*Notation 4.1.* If  $2 \leq n_0 \in \mathbb{N}$  and  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , we denote the path  $\mathbb{B}(t) \in \text{Class}(n, n_0, k_R, k_L)$  given in (22) by  $\mathbb{B}_{s, n_0, k_R, k_L}(t)$ . We denote  $\mathbb{A}$  given in (23) by  $\mathbb{A}_{s, n_0, k_R, k_L}$ , and  $\mathbb{V} \in \text{Class}(n, 1, n_0(k_R + 1) - 1, n_0(k_L + 1) - 1)$  given in Notation 4.10 associated to  $\mathbb{A}_{s, n_0, k_R, k_L}$  by  $\mathbb{V}_{s, n_0, k_R, k_L}$ . Furthermore, we denote the path of the orthogonal projections  $\hat{G}_l(t)$  given in Lemma 4.14 by  $\hat{G}_{n_0, k_R, k_L, l}(t)$ . We also set

$$\tilde{m}(n_0, k_R, k_L) := \max \left\{ 2n_0^6(k_R + 1)(k_L + 1), 2l_{\mathbb{A}_{s, n_0, k_R, k_L}}, 2l_{\mathbb{V}_{s, n_0, k_R, k_L}}, l_{4, n_0, k_R, k_L}, l_{3, n_0, k_R, k_L} \right\}, \quad (90)$$

where  $l_{\mathbb{A}, n_0, k_R, k_L}$ ,  $l_{4, n_0, k_R, k_L}$ ,  $l_{3, n_0, k_R, k_L}$  are the numbers  $l_{\mathbb{A}}$ ,  $l_4$ ,  $l_3$  given in Lemma 4.5, Lemma 4.15, Lemma 4.13, respectively. The number  $l_{\mathbb{V}, s, n_0, k_R, k_L}$  is the one defined in Definition 1.13 [O1] for  $\mathbb{V}_{s, n_0, k_R, k_L} \in \text{Class A}$ . For  $n_0 = 1$  and each  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , we fix some  $\mathbb{V}_{s, 1, k_R, k_L} \in \text{Class}(n, 1, k_R, k_L)$  and set  $\mathbb{B}_{s, 1, k_R, k_L}(t) := \mathbb{V}_{s, 1, k_R, k_L}$ ,  $t \in [0, 1]$ . We also set  $\tilde{m}(1, k_R, k_L) := 2(k_R + 1)(k_L + 1)$ .

**Proof of Theorem 1.8.** Let  $n_0, n'_0 \in \mathbb{N}$ ,  $k_R, k_L, k'_R, k'_L \in \mathbb{N} \cup \{0\}$ ,  $\mathbb{B} \in \text{Class}(n, n_0, k_R, k_L)$ , and  $\mathbb{B}' \in \text{Class}(n, n'_0, k'_R, k'_L)$ . Let  $m \geq 2l_{\mathbb{B}}$ , and  $m' \geq 2l_{\mathbb{B}'}$ . From [BMNS] and Theorem 1.18 (vi) of [O1], if  $H_{\Phi_{m, \mathbb{B}}} \simeq_{II} H_{\Phi_{m', \mathbb{B}'}}$ , we have  $n_0(k_R + 1) = n'_0(k'_R + 1)$ , and  $n_0(k_L + 1) = n'_0(k'_L + 1)$ .

Now we show the converse. Set  $m_0 := \max\{m, m', \tilde{m}(n_0, k_R, k_L), \tilde{m}(n'_0, k'_R, k'_L)\}$ . We consider the following sequence of paths.

1. By the same argument as Lemma 4.2 of [BO], by considering a path  $\Phi(X; t) := (1 - t)\Phi_{m, \mathbb{B}}(X) + t\Phi_{m_0, \mathbb{B}}(X)$ , we obtain  $H_{\Phi_{m, \mathbb{B}}} \simeq_I H_{\Phi_{m_0, \mathbb{B}}}$ . For  $N$  large enough, the kernel of  $(H_{\Phi_{m, \mathbb{B}}})_{[0, N-1]}$  and  $(H_{\Phi_{m_0, \mathbb{B}}})_{[0, N-1]}$  coincide. From this, Lemma 4.22, and Theorem 1.18 of [O1] we see that  $(\Phi_{m, \mathbb{B}}, \Phi_{m_0, \mathbb{B}})$  satisfy *Condition 8*.
2. As  $\mathbb{B}, \mathbb{B}_{s, n_0, k_R, k_L}(1) \in \text{Class}(n, n_0, k_R, k_L)$ , we have  $H_{\Phi_{m_0, \mathbb{B}}} \simeq_I H_{\Phi_{m_0, \mathbb{B}_{s, n_0, k_R, k_L}(1)}}$  from Proposition 3.1. By the proof of Proposition 3.1, the path connecting  $\Phi_{m_0, \mathbb{B}}$  and  $\Phi_{m_0, \mathbb{B}_{s, n_0, k_R, k_L}(1)}$  is given by  $\Phi_{m_0, \mathbb{B}(t)}$  with  $\mathbb{B}(t) \in \text{Class}(n, n_0, k_R, k_L)$ , a path of positive frustration free translation invariant finite range interactions satisfying *Condition 6*.
3. (If  $n_0 \geq 2$ .) By Lemma 4.34, the paths  $\{\hat{\xi}_{\alpha, \beta, a, b}^{(l)}\}$  given by Lemma 4.14 satisfies the *Condition 5* with  $m' = \max\{l_{4, n_0, k_R, k_L}, 2n_0^6(k_R + 1)(k_L + 1)\} \leq m_0$ . From Lemma 2.3, this implies  $H_{\Phi_{m_0, \mathbb{B}_{s, n_0, k_R, k_L}(1)}} = H_{\Phi_{\tilde{G}_{n_0, k_R, k_L, m_0(1)}}} \simeq_I H_{\Phi_{\tilde{G}_{n_0, k_R, k_L, m_0(0)}}}$ . The path connecting them is given by  $\Phi(t) = \Phi_{\tilde{G}_{n_0, k_R, k_L, m_0(t)}}$ , a path of positive frustration free translation invariant finite range interactions satisfying *Condition 6*.
4. (If  $n_0 \geq 2$ .) By Lemma 4.17, we have  $H_{\Phi_{\tilde{G}_{n_0, k_R, k_L, m_0(0)}}} \simeq_{II} H_{\Phi_{m_0, \mathbb{A}_{s, n_0, k_R, k_L}}}$  via the path  $\Phi(X; t) := (1 - t)\Phi_{\tilde{G}_{n_0, k_R, k_L, m_0(0)}}(X) + t\Phi_{m_0, \mathbb{A}_{s, n_0, k_R, k_L}}(X)$ . From Lemma 4.25, the pair of the interactions  $(\Phi_{\tilde{G}_{n_0, k_R, k_L, m_0(0)}}, \Phi_{m_0, \mathbb{A}_{s, n_0, k_R, k_L}})$  satisfies *Condition 8*.
5. (If  $n_0 \geq 2$ .) By Notation 4.10, we have  $H_{\Phi_{m_0, \mathbb{A}_{s, n_0, k_R, k_L}}} \simeq_{II} H_{\Phi_{m_0, \mathbb{V}_{s, n_0, k_R, k_L}}}$ , via the path  $\Phi(X; t) := (1 - t)\Phi_{m_0, \mathbb{A}_{s, n_0, k_R, k_L}}(X) + t\Phi_{m_0, \mathbb{V}_{s, n_0, k_R, k_L}}(X)$ . From Lemma 4.26, the pair of the interactions  $(\Phi_{m_0, \mathbb{A}_{s, n_0, k_R, k_L}}, \Phi_{m_0, \mathbb{V}_{s, n_0, k_R, k_L}})$  satisfies *Condition 8*.

Hence we obtain  $H_{\Phi_{m, \mathbb{B}}} \simeq_{II} H_{\Phi_{m_0, \mathbb{V}_{s, n_0, k_R, k_L}}}$ . Recall that we set  $\mathbb{B}_{s, 1, k_R, k_L}(1) := \mathbb{V}_{s, 1, k_R, k_L}$ , for  $n_0 = 1$ . Parallel argument implies  $H_{\Phi_{m', \mathbb{B}'}} \simeq_{II} H_{\Phi_{m_0, \mathbb{V}_{s, n'_0, k'_R, k'_L}}}$ . If  $n_0(k_R + 1) = n'_0(k'_R + 1)$ , and  $n_0(k_L + 1) = n'_0(k'_L + 1)$ , then we have  $\mathbb{V}_{s, n_0, k_R, k_L}, \mathbb{V}_{s, n'_0, k'_R, k'_L} \in \text{Class}(n, 1, n_0(k_R + 1) - 1, n_0(k_L + 1) - 1) = \text{Class}(n, 1, n'_0(k'_R + 1) - 1, n'_0(k'_L + 1) - 1)$ . Therefore we have  $H_{\Phi_{m_0, \mathbb{V}_{s, n_0, k_R, k_L}}} \simeq_I H_{\Phi_{m_0, \mathbb{V}_{s, n'_0, k'_R, k'_L}}}$  from Proposition 3.1. This completes the proof.  $\square$

## 5 Bulk classification

In this section we prove Theorem 1.12. First, recall that gap of local Hamiltonians imply that of bulk Hamiltonians.



**Lemma 5.1.** *Let  $\Phi$  be a translation invariant finite range interaction on  $\mathcal{A}_{\mathbb{Z}}$ . Assume that the Hamiltonian  $H_{\Phi}$  is gapped with respect to the open boundary condition. Let  $\gamma > 0$  be a lower bound of the gap. Then for any  $\omega \in \mathcal{S}_{\mathbb{Z}}(H_{\Phi})$ , we have*

$$\sigma(H_{\omega, \alpha_{\Phi}}) \setminus \{0\} \subset [\gamma, \infty).$$

**Proof.** The proof is the same as that of [KN], and [SS].  $\square$

As an immediate consequence of Lemma 5.1, we obtain the following.

**Lemma 5.2.** *Let  $H_0, H_1$  be Hamiltonians associated with interactions  $\Phi_0, \Phi_1 \in \mathcal{J}$  that are gapped with respect to the open boundary conditions. Assume that  $H_0$  and  $H_1$  are type I- $C^1$ -equivalent. Let  $m \in \mathbb{N}$ ,  $\Phi : [0, 1] \rightarrow \mathcal{J}_m$ , and  $\gamma > 0$  be the corresponding path of interactions and the uniform lower bound of the gap in Definition 1.2. Then, for any  $t \in [0, 1]$  and  $\omega_t \in \mathcal{S}_{\mathbb{Z}}(H_{\Phi(t)})$ , we have*

$$\sigma(H_{\omega_t, \alpha_{\Phi(t)}}) \setminus \{0\} \subset [\gamma, \infty).$$

## 5.1 Frustration free Hamiltonians and their bulk ground states

To check that some interaction  $\Phi$  belongs to  $\mathcal{J}_B$ , we have to consider not only elements in  $\mathcal{S}_{\mathbb{Z}}(H_{\Phi})$  but all elements in  $\mathfrak{B}_{\Phi}$ . For the interactions that is considered in this paper, these set coincides. Recall Notation 4.18. The proof of the following Lemma is exactly the same as that of Lemma 3.10 [O1].

**Lemma 5.3.** *Let  $H_{\Phi}$  be a frustration free Hamiltonian given by a positive interaction  $\Phi \in \mathcal{J}$ . Then we have  $\widetilde{\mathcal{S}}_{\mathbb{Z}, \mathbb{Z}}(H_{\Phi}) = \mathcal{S}_{\mathbb{Z}}(H_{\Phi})$ . If  $\omega$  is a unique element in  $\mathcal{S}_{\mathbb{Z}}(H_{\Phi})$ , it is pure.*

**Lemma 5.4.** *Let  $H_{\Phi}$  be a frustration free Hamiltonian given by a positive interaction  $\Phi \in \mathcal{J}$ . Let  $\omega$  be an element in  $\mathcal{S}_{\mathbb{Z}}(H_{\Phi})$  and  $(\mathcal{H}, \pi, \Omega)$  its GNS triple. Then for any unit vector  $\xi \in \ker H_{\omega, \alpha_{\Phi}}$ ,  $\omega_{\xi} := \langle \xi, \pi(\cdot) \xi \rangle$  defines a state in  $\mathcal{S}_{\mathbb{Z}}(H_{\Phi})$ .*

**Proof.** We first claim that

$$\left\| \left( \pi \left( (H_{\Phi})_{[a, b]} \right) \right)^{\frac{1}{2}} \eta \right\| \leq \left\| H_{\omega, \alpha_{\Phi}}^{\frac{1}{2}} \eta \right\|, \quad (91)$$

for all  $\eta$  in the domain of  $(H_{\omega, \alpha_{\Phi}}^{\frac{1}{2}})$  and  $a \leq b$ . To see this, we consider arbitrary  $a \leq b$ , finite interval  $I = [c, d]$  with  $c \leq a$ ,  $b \leq d$ , and  $A \in \mathcal{A}_I$ . Let  $R \in \mathbb{N}$  be the interaction length of  $\Phi$ . Then we have  $\delta_{\Phi}(A) = i \left[ (H_{\Phi})_{[c-R, d+R]}, A \right]$ , by the definition of  $\alpha_{\Phi}$ . From this, we have

$$\begin{aligned} \langle \pi(A) \Omega, H_{\omega, \alpha_{\Phi}} \pi(A) \Omega \rangle &= -i \omega(A^* \delta_{\Phi}(A)) = \omega \left( A^* (H_{\Phi})_{[c-R, d+R]} A \right) - \omega \left( A^* A \left( (H_{\Phi})_{[c-R, d+R]} \right) \right) \\ &= \omega \left( A^* (H_{\Phi})_{[c-R, d+R]} A \right) \geq \omega \left( A^* (H_{\Phi})_{[a, b]} A \right). \end{aligned}$$

The third equality is due to the frustration freeness of  $\Phi$  and that  $\omega \in \mathcal{S}_{\mathbb{Z}}(H_{\Phi}) = \widetilde{\mathcal{S}}_{\mathbb{Z}, \mathbb{Z}}(H_{\Phi})$ , due to Lemma 5.3. The inequality is by the positivity of  $\Phi$ . As  $\pi(\mathcal{A}_{\mathbb{Z}}^{\text{loc}}) \Omega$  is a core of  $H_{\omega, \alpha_{\Phi}}$ , this prove the claim.

By (91), for any unit vector  $\xi \in \ker H_{\omega, \alpha_{\Phi}}$ , and finite interval  $[a, b]$ , we have  $\pi \left( (H_{\Phi})_{[a, b]} \right) \xi = 0$ . This means  $\omega_{\xi} \in \widetilde{\mathcal{S}}_{\mathbb{Z}, \mathbb{Z}}(H_{\Phi}) = \mathcal{S}_{\mathbb{Z}}(H_{\Phi})$ .  $\square$

**Lemma 5.5.** Let  $\Phi_0, \Phi_1 \in \mathcal{J}$  be positive interactions. Assume that the pair  $(\Phi_0, \Phi_1)$  satisfies the Condition 7. Let  $\omega_\infty$  be the state in 4 of Condition 7. For each  $s \in [0, 1]$  we define  $\Phi(s) \in \mathcal{J}$  by

$$\Phi(X; s) := (1 - s)\Phi_0(X) + s\Phi_1(X), \quad X \in \mathfrak{S}_{\mathbb{Z}}. \quad (92)$$

Then, for each  $s \in (0, 1)$ ,  $\omega_\infty$  is the unique  $\alpha_{\Phi(s)}$ -ground state.

**Proof.** *Step 1.* Note that for any  $s \in (0, 1)$ , we have  $\omega_\infty \in \widetilde{\mathcal{S}_{\mathbb{Z}, \mathbb{Z}}}(H_{\Phi(s)})$ . This is because

$$\omega_\infty(\Phi(X; s)) = (1 - s)\omega_\infty(\Phi_0(X)) + s\omega_\infty(\Phi_1(X)) = 0, \quad X \in \mathfrak{S}_{\mathbb{Z}}$$

due to the frustration freeness of  $H_{\Phi_0}$ ,  $H_{\Phi_1}$  and Lemma 5.3. Therefore,  $H_{\Phi(s)}$  is frustration free and  $\omega_\infty \in \mathcal{S}_{\mathbb{Z}}(H_{\Phi(s)})$  for all  $s \in (0, 1)$ . In particular,  $\omega_\infty$  is an  $\alpha_{\Phi(s)}$ -ground state on  $\mathcal{A}_{\mathbb{Z}}$  for all  $s \in (0, 1)$ . Furthermore,  $\omega_\infty$  is pure, because it is the unique element in  $\mathcal{S}_{\mathbb{Z}}(H_{\Phi_0})$ , (Lemma 5.3).

*Step 2.* We claim that for any  $s \in (0, 1)$ , any pure  $\alpha_{\Phi(s)}$ -ground state  $\omega$  on  $\mathcal{A}_{\mathbb{Z}}$  is quasi-equivalent to  $\omega_\infty$ . Let  $s \in (0, 1)$  and  $\omega$  be an arbitrary pure  $\alpha_{\Phi(s)}$ -ground state on  $\mathcal{A}_{\mathbb{Z}}$  and  $(\mathcal{H}, \pi, \Omega)$  the GNS triple of  $\omega$ .

As  $\Phi_0, \Phi_1$  are of finite range, there exists an  $R > 0$  such that  $\Phi_0(X) = \Phi_1(X) = 0$  if the diameter of  $X$  is larger than  $R$ . Let  $\gamma > 0$  and  $N_0 \in \mathbb{N}$  be numbers satisfying  $\gamma \leq \inf_{N \geq N_0} \inf \left( \sigma \left( (H_{\Phi_1})_{[0, N-1]} \right) \setminus \{0\} \right)$ , from 2. of Condition 7.

First we claim

$$\sum_{X \in \mathfrak{S}_{\mathbb{Z}}} \omega(\Phi(X; s)) < \infty. \quad (93)$$

To see this, recall the characterization of the ground state in Theorem 6.2.52 of [BR96]. For each  $M \in \mathbb{N}$ , we choose a state  $\sigma_M$  on  $\mathcal{A}_{[-M, M]}$  whose support is in the kernel of  $(H_{\Phi_1})_{[-M, M]}$ . We consider the state  $\omega'_M := \sigma_M \otimes \omega|_{\mathcal{A}_{\mathbb{Z} \setminus [-M, M]}}$  on  $\mathcal{A}_{\mathbb{Z}}$ . Note that  $\omega'_M$  and  $\omega$  coincide on  $\mathcal{A}_{\mathbb{Z} \setminus [-M, M]}$ . By Theorem 6.2.52 of [BR96], as  $\omega$  is an  $\alpha_{\Phi(s)}$ -ground state, we obtain

$$\begin{aligned} 0 &\leq \sum_{X: X \cap [-M, M] \neq \emptyset} \omega(\Phi(X; s)) \leq \sum_{X: X \cap [-M, M] \neq \emptyset} \omega'_M(\Phi(X; s)) = \sum_{X: X \cap [-M, M] \neq \emptyset} ((1 - s)\omega'_M(\Phi_0(X)) + s\omega'_M(\Phi_1(X))) \\ &= (1 - s)\sigma_M \left( (H_{\Phi_0})_{[-M, M]} \right) + s\sigma_M \left( (H_{\Phi_1})_{[-M, M]} \right) + \sum_{\substack{X \cap [-M, M] \neq \emptyset \\ X \cap [-M, M]^c \neq \emptyset}} \omega'_M(\Phi(X; s)). \end{aligned} \quad (94)$$

As the support of  $\sigma_M$  is in the kernel of  $(H_{\Phi_1})_{[-M, M]}$ , we have  $\sigma_M \left( (H_{\Phi_1})_{[-M, M]} \right) = 0$ . In order to estimate  $\sigma_M \left( (H_{\Phi_0})_{[-M, M]} \right)$ , set  $C_1 := \sum_{X: 0 \in X} \|\Phi_1(X)\|$ , and note that  $\left\| (H_{\Phi_0})_{[-M, M]} \right\| \leq C_1(2M + 1)$ . We then obtain

$$\begin{aligned} \sigma_M \left( (H_{\Phi_0})_{[-M, M]} \right) &= \sigma_M \left( (H_{\Phi_0})_{[-M, M]} (G_{[-M, M], 1} - G_{[-M, M], 0}) G_{[-M, M], 1} \right) + \sigma_M \left( (H_{\Phi_0})_{[-M, M]} G_{[-M, M], 0} G_{[-M, M], 1} \right) \\ &= \sigma_M \left( (H_{\Phi_0})_{[-M, M]} (G_{[-M, M], 1} - G_{[-M, M], 0}) G_{[-M, M], 1} \right) \\ &\leq \left\| (G_{[-M, M], 1} - G_{[-M, M], 0}) G_{[-M, M], 1} \right\| \left\| (H_{\Phi_0})_{[-M, M]} \right\| \leq C_1 \cdot Cr^{2M+1}(2M + 1). \end{aligned}$$

Here,  $C, r$  are constants given in 3. of Condition 7. Hence  $\sigma_M \left( (H_{\Phi_0})_{[-M, M]} \right)$  converges to 0 as  $M \rightarrow \infty$ .

By the translation invariance of  $\Phi_0, \Phi_1$  and the definition of  $R$ , for any  $M \geq R$ , we have

$$\sum_{\substack{X \cap [-M, M] \neq \emptyset \\ X \cap [-M, M]^c \neq \emptyset}} (1 - s)\|\Phi_0(X)\| + s\|\Phi_1(X)\| = \sum_{\substack{X \cap [-R, R] \neq \emptyset \\ X \cap [-R, R]^c \neq \emptyset}} (1 - s)\|\Phi_0(X)\| + s\|\Phi_1(X)\| =: C_2.$$

Substitute this and  $\lim_M \sigma_M \left( (H_{\Phi_0})_{[-M,M]} \right) = 0$ ,  $\sigma_M \left( (H_{\Phi_1})_{[-M,M]} \right) = 0$  to (94). Taking  $M \rightarrow \infty$  limit, we obtain  $0 \leq \sum_{X \in \mathbb{S}_\mathbb{Z}} \omega(\Phi(X; s)) \leq C_2 < \infty$ , proving the claim (93).

From this, there exists an  $M_0 \in \mathbb{N}$  with  $R \leq M_0$  such that

$$0 \leq s \sum_{X \subset [-M_0, M_0]^c} \omega(\Phi_1(X)) \leq \sum_{X \subset [-M_0, M_0]^c} \omega(\Phi(X; s)) < \frac{s\gamma}{4}. \quad (95)$$

By the spectral gap condition for  $\Phi_1$ , for any  $N \in \mathbb{N}$  with  $M_0 + N_0 \leq N$ , we have

$$\| (1 - G_{[-N, N] \setminus [-M_0, M_0], 1}) \Omega \|^2 \leq \gamma^{-1} \sum_{X \subset [-M_0, M_0]^c} \omega(\Phi_1(X)) < \frac{1}{4}. \quad (96)$$

Now we set for each  $N_0 + M_0 \leq N$ , a vector  $\xi_N := G_{[-N, N] \setminus [-M_0, M_0], 1} \Omega$ . The norm of these vectors are decreasing by the definition. From (96), we have  $\|\xi_N\| \geq \frac{1}{2}$  for each  $N_0 + M_0 \leq N$ . Therefore  $\|\xi_N\|$  converges to a strictly positive constant. Furthermore, for  $N' \leq N$ , by the definition, we have

$$\|\xi_N - \xi_{N'}\|^2 = \|\xi_N\|^2 + \|\xi_{N'}\|^2 - \langle \xi_N, \xi_{N'} \rangle - \langle \xi_{N'}, \xi_N \rangle = -\|\xi_N\|^2 + \|\xi_{N'}\|^2.$$

This and the fact that  $\|\xi_N\|$  converges means that  $\{\xi_N\}_N$  is a Cauchy sequence. Therefore, it has a limit  $\tilde{\xi} \in \mathcal{H}$ . From the bound  $\|\xi_N\| \geq \frac{1}{2}$ ,  $\tilde{\xi}$  is non-zero, and we define a vector  $\xi := \|\tilde{\xi}\|^{-1} \tilde{\xi}$  in  $\mathcal{H}$  and an  $\omega$ -normal state  $\omega_\xi := \langle \xi, \pi(\cdot) \xi \rangle$ . As  $\omega$  is pure,  $\omega_\xi$  and  $\omega$  are quasi-equivalent. From the definition of this  $\tilde{\xi}$ , for any  $X \in \mathbb{S}_\mathbb{Z}$ , with  $X \cap [-M_0, M_0] = \emptyset$ , we have  $\omega_\xi(\Phi_1(X)) = 0$ . Hence  $\omega_\xi$  belongs to  $\mathcal{S}_{\mathbb{Z}, [-M_0, M_0]^c}(H_{\Phi_1})$ . By 5. of the assumption, this implies that  $\omega_\xi$  and  $\omega_\infty$  are quasi-equivalent. Combined with the quasi-equivalence of  $\omega$  and  $\omega_\xi$ , we conclude that  $\omega$  and  $\omega_\infty$  are quasi-equivalent, proving the claim of *Step 2*.

*Step 3.* We claim that for any  $s \in (0, 1)$ , if  $\omega$  is a pure  $\alpha_{\Phi(s)}$ -ground state, then we have  $\omega = \omega_\infty$ . Let  $(\mathcal{H}_\infty, \pi_\infty, \Omega_\infty)$  be the GNS triple of  $\omega_\infty$ . As  $\omega_\infty$  is pure, we have  $\pi_\infty(\mathcal{A}_\mathbb{Z})'' = \mathcal{B}(\mathcal{H}_\infty)$ . From *Step 2*,  $\omega$  is  $\omega_\infty$ -normal, hence represented by a density matrix on  $\mathcal{H}$ . As furthermore  $\omega$  is pure, the density matrix is a one rank projection onto  $\mathbb{C}\xi$  for some unit vector  $\xi \in \mathcal{H}_\infty$  i.e.,  $\omega = \omega_\xi = \langle \xi, \pi_\infty(\cdot) \xi \rangle$ . By the  $\alpha_{\Phi(s)}$ -invariance of  $\omega$  and  $\pi_\infty(\mathcal{A}_\mathbb{Z})'' = \mathcal{B}(\mathcal{H}_\infty)$ , we have  $e^{itH_{\omega_\infty, \alpha_{\Phi(s)}}}(|\xi\rangle\langle\xi|)e^{-itH_{\omega_\infty, \alpha_{\Phi(s)}}} = |\xi\rangle\langle\xi|$  for all  $t \in \mathbb{R}$ . From this and the strong continuity of the one parameter group of unitaries  $e^{itH_{\omega_\infty, \alpha_{\Phi(s)}}}$ ,  $\xi$  is an eigenvector of  $H_{\omega_\infty, \alpha_{\Phi(s)}}$  with some eigenvalue  $\lambda$ . From the positivity of  $H_{\omega_\infty, \alpha_{\Phi(s)}}$ , we have  $\lambda \geq 0$ . The triple  $(\mathcal{H}_\infty, \pi_\infty, \xi)$  is the GNS triple of  $\omega$  because of  $\pi_\infty(\mathcal{A}_\mathbb{Z})'' = \mathcal{B}(\mathcal{H}_\infty)$ , and  $H_{\omega_\infty, \alpha_{\Phi(s)}} - \lambda$  satisfies

$$e^{it(H_{\omega_\infty, \alpha_{\Phi(s)}} - \lambda)} \pi_\infty(A) \xi = \pi_\infty(\alpha_{\Phi(s), t}(A)) \xi, \quad A \in \mathcal{A}_\mathbb{Z}, \quad t \in [0, 1].$$

Hence we have  $H_{\omega, \alpha_{\Phi(s)}} = H_{\omega_\infty, \alpha_{\Phi(s)}} - \lambda$ . As this operator have to be positive, we have

$$0 \leq \langle \Omega_\infty, H_{\omega, \alpha_{\Phi(s)}} \Omega_\infty \rangle = \langle \Omega_\infty, H_{\omega_\infty, \alpha_{\Phi(s)}} \Omega_\infty \rangle - \lambda = -\lambda.$$

Hence we conclude  $\lambda = 0$ , and  $\xi$  is in the kernel of  $H_{\omega_\infty, \alpha_{\Phi(s)}}$ . From Lemma 5.4 we obtain that  $\omega = \omega_\xi \in \mathcal{S}_\mathbb{Z}(H_{\Phi(s)}) = \widetilde{\mathcal{S}_{\mathbb{Z}, \mathbb{Z}}}(H_{\Phi(s)}) \subset \widetilde{\mathcal{S}_{\mathbb{Z}, \mathbb{Z}}}(H_{\Phi_1}) = \mathcal{S}_\mathbb{Z}(H_{\Phi_1}) = \{\omega_\infty\}$ , i.e.,  $\omega = \omega_\infty$ .

*Step 4.* Let  $s \in (0, 1)$ . As the set of all  $\alpha_{\Phi(s)}$ -ground states  $\mathfrak{B}_{\Phi(s)}$  is a non-empty  $wk^*$ -compact convex set, by Krein-Milman theorem,  $\mathfrak{B}_{\Phi(s)}$  is the convex closed hull of its extremal points. Any extremal point of is pure  $\mathfrak{B}_{\Phi(s)}$  (Theorem 5.3.37 of [BR96]). Therefore, from *Step 3*,  $\omega_\infty$  is the unique extremal point of  $\mathfrak{B}_{\Phi(s)}$ . This means  $\omega_\infty$  is the unique  $\alpha_{\Phi(s)}$ -ground state.  $\square$

We obtain the following corollary as a special case of the previous Lemma.

**Corollary 5.6.** *Let  $H_\Phi$  be a frustration free Hamiltonian given by a positive interaction  $\Phi \in \mathcal{J}$ , and assume that  $H_\Phi$  is gapped with respect to the open boundary conditions. Assume  $\Phi$  satisfies Condition 6. Then, the  $\alpha_\Phi$ -ground state is unique.*

**Proof.** Set  $\Phi_0 = \Phi_1 = \Phi$  in Lemma 5.5.  $\square$

## 5.2 Bulk equivalence

In this subsection, we derive sufficient conditions for the bulk equivalence.

**Lemma 5.7.** *Let  $H_0, H_1$  be gapped Hamiltonians with respect to the open boundary conditions, associated with interactions  $\Phi_0, \Phi_1 \in \mathcal{J}$ . Assume that  $H_{\Phi_0}$  and  $H_{\Phi_1}$  are type I- $C^1$ -equivalent, and let  $\Phi$  be the corresponding path of interactions. Assume that for each  $s \in [0, 1]$ ,  $H_{\Phi(s)}$  is frustration-free and  $\Phi(s)$  satisfies Condition 6. Then we have  $H_{\Phi_0} \simeq_B H_{\Phi_1}$ ,*

**Proof.** By Corollary 5.6, there exists a unique  $\alpha_{\Phi(s)}$ -ground state  $\omega_s$  for each  $s \in [0, 1]$ . As  $\mathcal{S}_\mathbb{Z}(H_{\Phi(s)})$  is non-empty, this means  $\omega_s \in \mathcal{S}_\mathbb{Z}(H_{\Phi(s)})$ . Let  $\gamma > 0$  be the uniform lower bound of the gap in Definition 1.2. By Lemma 5.2, we then have

$$\sigma(H_{\omega_s, \alpha_{\Phi(s)}}) \setminus \{0\} \subset [\gamma, \infty).$$

By the uniqueness of the ground state, 0 is a non-degenerate eigenvalue of  $H_{\omega_s, \alpha_{\Phi(s)}}$ , for each  $s \in [0, 1]$ . This proves  $\Phi(s) \in \mathcal{J}_B$ , and  $H_{\Phi_0} \simeq_B H_{\Phi_1}$ .  $\square$

**Lemma 5.8.** *Let  $\Phi_0, \Phi_1 \in \mathcal{J}$  be positive interactions. Assume that the pair  $(\Phi_0, \Phi_1)$  satisfies the Condition 8. Then we have  $\Phi_0, \Phi_1 \in \mathcal{J}_B$ . Furthermore, for each  $s \in [0, 1]$  we define  $\Phi(s) \in \mathcal{J}$  by (92). Then we have  $H_{\Phi_0} \simeq_B H_{\Phi_1}$  with respect to this path.*

**Proof.** First we note that for any  $s \in [0, 1]$ ,  $\omega_\infty$  in 4. of Condition 7 is the unique  $\alpha_{\Phi(s)}$ -ground state.: By Lemma 5.5,  $\omega_\infty$  is the unique  $\alpha_{\Phi(s)}$ -ground state for each  $s \in (0, 1)$ . For  $s = 0, 1$ , we obtain it by Corollary 5.6.

Let  $(\mathcal{H}_\infty, \pi_\infty, \Omega_\infty)$  be the GNS triple of  $\omega_\infty$ . We have  $\pi_\infty(\mathcal{A}_\mathbb{Z})'' = \mathcal{B}(\mathcal{H}_\infty)$  because  $\omega_\infty$  is pure. Therefore, for any  $s \in [0, 1]$ , any unit vector  $\eta$  in the kernel of  $H_{\omega_\infty, \alpha_{\Phi(s)}}$  orthogonal to  $\Omega_\infty$  gives an  $\alpha_{\Phi(s)}$ -ground state  $\langle \eta, \pi_\infty(\cdot)\eta \rangle$  which is different from  $\omega_\infty$ . Recall that our  $\omega_\infty$  is the unique  $\alpha_{\Phi(s)}$ -ground state. Therefore, we have  $\ker H_{\omega_\infty, \alpha_{\Phi(s)}} = \mathbb{C}\Omega_\infty$  for any  $s \in [0, 1]$ .

From the assumption and Lemma 5.1, there exists  $\gamma > 0$  such that

$$\sigma(H_{\omega_\infty, \alpha_{\Phi_i}}) \setminus \{0\} \subset [\gamma, \infty), \quad i = 0, 1. \quad (97)$$

In particular, we have  $\Phi_0, \Phi_1 \in \mathcal{J}_B$ . In order to complete the proof, it suffices to show

$$\sigma(H_{\omega_\infty, \alpha_{\Phi(s)}}) \setminus \{0\} \subset [\gamma, \infty) \quad (98)$$

for any  $s \in [0, 1]$ . To prove this, it suffices to show that the spectral projection of  $H_{\omega_\infty, \alpha_{\Phi(s)}}$  corresponding to  $[0, \lambda]$  is equal to the orthogonal projection onto  $\mathbb{C}\Omega_\infty$ , for all  $0 < \lambda < \gamma$ . We assume this is not the case, and show a contradiction. Suppose that there exists a  $0 < \lambda < \gamma$  and a unit vector  $\xi \in \mathcal{H}_\infty$  such that orthogonal to  $\Omega_\infty$  and  $\text{Proj}[H_{\omega_\infty, \alpha_{\Phi(s)}} \in [0, \lambda]]\xi = \xi$ . (Recall the notation  $\text{Proj}$  in Appendix A [O1].) As  $\xi$  is in the domain of  $H_{\omega_\infty, \alpha_{\Phi(s)}}$  and  $\pi_\infty(\mathcal{A}_\mathbb{Z}^{\text{loc}})\Omega_\infty$  is a core of  $H_{\omega_\infty, \alpha_{\Phi(s)}}$  [BR96], there exists a net  $\{A_\beta\}_\beta$  in  $\mathcal{A}_\mathbb{Z}^{\text{loc}}$  such that

$$\lim_\beta \|\pi_\infty(A_\beta)\Omega_\infty - \xi\| = 0, \quad \lim_\beta \|H_{\omega_\infty, \alpha_{\Phi(s)}}\pi_\infty(A_\beta)\Omega_\infty - H_{\omega_\infty, \alpha_{\Phi(s)}}\xi\| = 0. \quad (99)$$

Note that  $\omega_\infty(A_\beta) = \langle \Omega_\infty, \pi_\infty(A_\beta) \Omega_\infty \rangle \rightarrow \langle \Omega_\infty, \xi \rangle = 0$  from this equation. Therefore, setting  $\tilde{A}_\beta := A_\beta - \omega_\infty(A_\beta)$ , we obtain

$$\lim_{\beta} \left\| \pi_\infty(\tilde{A}_\beta) \Omega_\infty - \xi \right\| = 0, \quad \lim_{\beta} \left\| H_{\omega_\infty, \alpha_{\Phi(s)}} \pi_\infty(\tilde{A}_\beta) \Omega_\infty - H_{\omega_\infty, \alpha_{\Phi(s)}} \xi \right\| = 0. \quad (100)$$

By the definition,  $\pi_\infty(\tilde{A}_\beta) \Omega_\infty$  and  $\Omega_\infty$  are orthogonal. Recall that we have  $\ker H_{\omega_\infty, \alpha_{\Phi_i}} = \mathbb{C} \Omega_\infty$   $i = 0, 1$ , and (97). From these facts,  $\pi_\infty(\tilde{A}_\beta) \Omega_\infty$  belongs to  $\text{Proj}[H_{\omega_\infty, \alpha_{\Phi_i}} \in [\gamma, \infty)] \mathcal{H}$ ,  $i = 0, 1$ . Now using the definition of the bulk Hamiltonian [BR96], we have

$$\begin{aligned} & \left\langle \pi_\infty(\tilde{A}_\beta) \Omega_\infty, H_{\omega_\infty, \alpha_{\Phi(s)}} \pi_\infty(\tilde{A}_\beta) \Omega_\infty \right\rangle = -i \left\langle \pi_\infty(\tilde{A}_\beta) \Omega_\infty, \pi_\infty(\delta_{\alpha_{\Phi(s)}}(\tilde{A}_\beta)) \Omega_\infty \right\rangle \\ & = -i \left\langle \pi_\infty(\tilde{A}_\beta) \Omega_\infty, \left( (1-s) \pi_\infty(\delta_{\alpha_{\Phi_0}}(\tilde{A}_\beta)) + s \pi_\infty(\delta_{\alpha_{\Phi_1}}(\tilde{A}_\beta)) \right) \Omega_\infty \right\rangle \\ & = (1-s) \left\langle \pi_\infty(\tilde{A}_\beta) \Omega_\infty, H_{\omega_\infty, \alpha_{\Phi_0}} \pi_\infty(\tilde{A}_\beta) \Omega_\infty \right\rangle + s \left\langle \pi_\infty(\tilde{A}_\beta) \Omega_\infty, H_{\omega_\infty, \alpha_{\Phi_1}} \pi_\infty(\tilde{A}_\beta) \Omega_\infty \right\rangle \\ & \geq ((1-s)\gamma + s\gamma) \left\| \pi_\infty(\tilde{A}_\beta) \Omega_\infty \right\|^2 = \gamma \left\| \pi_\infty(\tilde{A}_\beta) \Omega_\infty \right\|^2. \end{aligned}$$

Taking the  $\beta \rightarrow \infty$  limit, from the choice of  $\xi$ , we obtain

$$\lambda \geq \langle \xi, H_{\omega_\infty, \alpha_{\Phi(s)}} \xi \rangle \geq \gamma,$$

which contradict  $\lambda < \gamma$ . This proves (98) for all  $s \in [0, 1]$ .  $\square$

**Lemma 5.9.** *Let  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \boldsymbol{\lambda}, \mathbb{D}, \mathbb{G}, Y)$ . Let  $\boldsymbol{\omega}_{\mathbb{B}} \in \text{Prim}_1(n, n_0)$  be the  $n$ -tuple given by Lemma 3.2 of [O1]. Then we have  $H_{\Phi_{m, \mathbb{B}}} \simeq_B H_{\Phi_{m, \boldsymbol{\omega}_{\mathbb{B}}}}$ , for  $m \geq 2l_{\mathbb{B}}$ .*

**Proof.** For any  $l \geq l_{\mathbb{B}}$ , we have

$$\mathcal{K}_l(\boldsymbol{\omega}_{\mathbb{B}}) \otimes E_{00}^{(k_R, k_L)} = \left( \mathbb{I}_{n_0} \otimes E_{00}^{(k_R, k_L)} \right) \mathcal{K}_l(\mathbb{B}) \left( \mathbb{I}_{n_0} \otimes E_{00}^{(k_R, k_L)} \right) = M_{n_0} \otimes E_{00}^{(k_R, k_L)}.$$

From this, we have  $l_{\boldsymbol{\omega}_{\mathbb{B}}} \leq l_{\mathbb{B}}$ , and  $G_{N, \boldsymbol{\omega}_{\mathbb{B}}} \leq G_{N, \mathbb{B}}$  for all  $l_{\mathbb{B}} \leq N$ .

We apply Lemma 5.8 to  $(\Phi_0, \Phi_1) = (\Phi_{m, \mathbb{B}}, \Phi_{m, \boldsymbol{\omega}_{\mathbb{B}}})$  with  $m \geq 2l_{\mathbb{B}}$ . We check *Condition 7* for  $(\Phi_{m, \mathbb{B}}, \Phi_{m, \boldsymbol{\omega}_{\mathbb{B}}})$ . The first and second condition of *Condition 7* follows from Theorem 1.18 of [O1]. The third condition follows from  $G_{N, \boldsymbol{\omega}_{\mathbb{B}}} \leq G_{N, \mathbb{B}}$ ,  $l_{\mathbb{B}} \leq N$ . From this  $G_{N, \boldsymbol{\omega}_{\mathbb{B}}} \leq G_{N, \mathbb{B}}$  and the frustration freeness, we have  $\{\omega_{\boldsymbol{\omega}_{\mathbb{B}}, \infty}\} = \mathcal{S}_{\mathbb{Z}}(H_{\Phi_{m, \boldsymbol{\omega}_{\mathbb{B}}}}) \subset \mathcal{S}_{\mathbb{Z}}(H_{\Phi_{m, \mathbb{B}}}) = \{\omega_{\mathbb{B}, \infty}\}$ . This proves the fourth condition of *Condition 7*. The fifth condition of *Condition 7* follows from Lemma 4.22. Furthermore,  $(\Phi_{m, \mathbb{B}}, \Phi_{m, \boldsymbol{\omega}_{\mathbb{B}}})$  satisfies *Condition 8* with respect to (ii) by Theorem 1.18 of [O1] and Lemma 4.22. Hence we may apply Lemma 5.8 and obtain  $H_{\Phi_{m, \mathbb{B}}} \simeq_B H_{\Phi_{m, \boldsymbol{\omega}_{\mathbb{B}}}}$ .  $\square$

### 5.3 Parents Hamiltonian

In this subsection, we connect  $\mathcal{J}_{FB}$  to MPS Hamiltonians. The key ingredient is [M1].

**Lemma 5.10.** *Let  $\Phi \in \mathcal{J}_{FB}$ . Then there exist  $n_0 \in \mathbb{N}$  and  $\mathbf{v} \in \text{Prim}_1(n, n_0)$  such that*

$$H_\Phi \simeq_B H_{\Phi_{m, \mathbf{v}}}, \quad m \geq 2l_{\mathbf{v}}.$$

Furthermore, we have  $G_{N, \mathbf{v}} \leq G_{N, \Phi}$  for all  $N \in \mathbb{N}$ , where  $G_{N, \Phi}$  denotes the orthogonal projection onto  $\ker H_{\Phi, [0, N-1]}$ .

**Proof.** Let  $\omega$  be the unique  $\alpha_\Phi$ -ground state and  $(\mathcal{H}, \pi, \Omega)$  its GNS triple. As  $\omega$  is the unique  $\alpha_\Phi$ -ground state and  $H_\Phi$  is frustration free, we have  $\omega \in \tilde{\mathcal{S}}_{\mathbb{Z}, \mathbb{Z}}(H_\Phi) = \mathcal{S}_{\mathbb{Z}}(H_\Phi)$ . The uniqueness also implies  $\ker H_{\omega, \alpha_\Phi} = \mathbb{C}\Omega$ .

By Theorem 4.1 of [NS], the second condition of Definition 1.9 and  $\ker H_{\omega, \alpha_\Phi} = \mathbb{C}\Omega$  implies  $\lim_{N \rightarrow \infty} \omega(A \tau_{\pm N}(B)) = \omega(A) \omega(B)$ , for all  $A, B \in \mathcal{A}_{\mathbb{Z}}^{\text{loc}}$ . Therefore, we can apply Theorem 1.2 of [M1] and  $\omega$  is a pure finitely correlated state. Applying Theorem 1.5 of [FNW2], we see that  $\omega$  is right-generated by a minimal standard triple (see [FNW2] or [O2] Appendix C)  $(M_{n_0}, \mathbb{E}, \rho)$  where  $\mathbb{E}$  is given by an isometry  $V : \mathbb{C}^{n_0} \rightarrow \mathbb{C}^n \otimes \mathbb{C}^{n_0}$ , as  $\mathbb{E}(X) = V^* X V$ ,  $X \in M_n \otimes M_{n_0}$ . Furthermore, from Proposition 3.7 and Proposition 2.4 of [FNW2],  $\sigma(\mathbb{E}_{\mathbb{I}}) \cap \mathbb{T} = \{1\}$ , and 1 is a non-degenerate eigenvalue of  $\mathbb{E}_{\mathbb{I}}$ . The isometry  $V$  can be decomposed as  $V\chi = \sum_{\mu=1}^n \psi_\mu \otimes v_\mu^* \chi$ ,  $\chi \in \mathbb{C}^{n_0}$  with  $v_\mu \in M_{n_0}$ . With this notation, we have  $T_v = \mathbb{E}_{\mathbb{I}}$  and it is a unital CP map. The state  $\rho$  is faithful and  $T_v$ -invariant. Therefore,  $T_v$  is primitive and  $r_{T_v} = 1$ . Hence we have  $v \in \text{Prim}_1(n, n_0)$ .

We check that  $G_{l,v} \leq G_{l,\Phi}$  for all  $l \in \mathbb{N}$ . We claim that for any  $l \in \mathbb{N}$ , we have  $s(\omega|_{\mathcal{A}_{[0, l-1]}}) = G_{l,v}$ . For  $\xi \in \bigotimes_{i=0}^{l-1} \mathbb{C}^n$ , we have

$$\omega(|\xi\rangle\langle\xi|) = \rho \left( \left( \sum_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \langle \hat{\psi}_{\mu^{(l)}}, \xi \rangle \hat{v}_{\mu^{(l)}} \right) \left( \sum_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \langle \hat{\psi}_{\mu^{(l)}}, \xi \rangle \hat{v}_{\mu^{(l)}} \right)^* \right),$$

by a straightforward calculation. From this and the faithfulness of  $\rho$ , we see that  $\omega(|\xi\rangle\langle\xi|) = 0$  if and only if  $\sum_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \langle \hat{\psi}_{\mu^{(l)}}, \xi \rangle \hat{v}_{\mu^{(l)}} = 0$ . By the definition of  $\Gamma_{l,v}^{(R)}$ ,  $\sum_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \langle \hat{\psi}_{\mu^{(l)}}, \xi \rangle \hat{v}_{\mu^{(l)}} = 0$  if and only if  $\xi \in \mathcal{G}_{l,v}^\perp$ . This proves the claim. As we have  $\omega \in \tilde{\mathcal{S}}_{\mathbb{Z}, \mathbb{Z}}(H_\Phi)$ ,  $\omega(H_{\Phi, [0, l-1]}) = 0$  holds for each  $l \in \mathbb{N}$ . From the claim above, it means that  $G_{l,v} = s(\omega|_{\mathcal{A}_{[0, l-1]}}) \leq G_{l,\Phi}$  for all  $l \in \mathbb{N}$ .

Now we would like to show  $H_\Phi \simeq_B H_{\Phi_{m,v}}$  if  $m \geq 2l_v$ . We apply Lemma 5.8 to  $(\Phi_0, \Phi_1) = (\Phi, \Phi_{m,v})$ . We have to check *Condition 8* for  $(\Phi, \Phi_{m,v})$ . 1.2. of *Condition 7* is from  $\Phi \in \mathcal{J}_{FB}$  and Theorem 1.18 of [O1] with  $v \in \text{Prim}_1(n, n_0)$ . The third condition follows from  $G_{l,v} \leq G_{l,\Phi}$ ,  $l \in \mathbb{N}$ . As  $\omega$  is generated by  $v$ , we have  $\mathcal{S}_{\mathbb{Z}}(H_{\Phi_{m,v}}) = \{\omega_{v,\infty}\} = \{\omega\} = \mathcal{S}_{\mathbb{Z}}(H_\Phi)$ , and 4 of *Condition 7* follows. 5. of *Condition 7* follows from Lemma 4.22. As our  $\Phi$  belongs to  $\mathcal{J}_{FB}$ , (i) of *Condition 8* holds. Hence we may apply Lemma 5.8, and obtain  $H_\Phi \simeq_B H_{\Phi_{m,v}}$ .  $\square$

## 5.4 Proof of Theorem 1.12

Let  $\Phi_0, \Phi_1 \in \mathcal{J}_{FB}$ . By Lemma 5.10, we obtain  $m_i, n_{0,i} \in \mathbb{N}$  and  $v_i \in \text{Prim}_1(n, n_{0,i})$  with  $m_i \geq 2l_{v_i}$ , for each  $i = 0, 1$ , such that  $H_{\Phi_i} \simeq_B H_{\Phi_{m_i, v_i}}$ . Set  $k_0 := \max\{m_i, \tilde{m}(n_{0,i}, 0, 0)\}_{i=0,1}$ . (Recall (90).) We consider the sequence of paths 1.-5. given in the proof of Theorem 1.8, for  $m = m_i$ ,  $\mathbb{B} = v_i$ ,  $i = 0, 1$ . For the paths 1., 4., 5, *Condition 8* holds that we may apply Lemma 5.8 to get  $\simeq_B$ . For 2., 3, frustration-freeness of the path, *Condition 6* and  $\simeq_I$  implies  $\simeq_B$  by Lemma 5.7. Hence we obtain  $H_{\Phi_i} \simeq_B H_{\Phi_{m_i, v_i}} \simeq_B H_{\Phi_{k_0, v_{s, n_{0,i}, 0, 0}}}$ ,  $i = 0, 1$ . By Lemma 5.9, we obtain  $H_{\Phi_{k_0, v_{s, n_{0,i}, 0, 0}}} \simeq_B H_{\Phi_{k_0, \omega_{v_{s, n_{0,i}, 0, 0}}}}$ ,  $i = 0, 1$ . Note that  $\omega_{v_{s, n_{0,i}, 0, 0}} \in \text{Prim}_1(n, 1)$ ,  $i = 0, 1$ . Therefore, as in path 2. of Theorem 1.8 we obtain  $H_{\Phi_{k_0, \omega_{v_{s, n_{0,i}, 0, 0}}}} \simeq_B H_{\Phi_{k_0, \omega_{v_{s, n_{0,1}, 0, 0}}}}$ . Hence we obtain

$$H_{\Phi_0} \simeq_B H_{\Phi_{m_0, v_0}} \simeq_B H_{\Phi_{k_0, v_{s, n_{0,0}, 0, 0}}} \simeq_B H_{\Phi_{k_0, \omega_{v_{s, n_{0,0}, 0, 0}}}} \simeq_B H_{\Phi_{k_0, \omega_{v_{s, n_{0,1}, 0, 0}}}}} \simeq_B H_{\Phi_{k_0, v_{s, n_{0,1}, 0, 0}}} \simeq_B H_{\Phi_{m_1, v_1}} \simeq_B H_{\Phi_1},$$

proving the Theorem.

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## A $C^\infty$ -path of linear independent vectors

**Lemma A.1.** *Let  $k, m \in \mathbb{N}$  with  $m \leq k$ . Let  $\zeta_i : [0, 1] \rightarrow \mathbb{C}^k$ ,  $i = 1, \dots, m$  be  $C^\infty$ -maps. For each  $t \in [0, 1]$ , let  $P(t)$  be the orthogonal projection onto  $\text{span}\{\zeta_i(t)\}_{i=1}^m$ . Suppose that for each  $t \in [0, 1]$ , the vectors  $\{\zeta_i(t)\}_{i=1}^m$  are linearly independent. Then there exist  $C^\infty$ -maps  $\eta_i : [0, 1] \rightarrow \mathbb{C}^k$ ,  $i = 1, \dots, m$  such that  $\{\eta_i(t)\}_{i=1}^m$  is a CONS of  $P(t)\mathbb{C}^k$ , for each  $t \in [0, 1]$ . In particular, the map  $[0, 1] \ni t \mapsto P(t) \in \mathbb{M}_k$  is  $C^\infty$ .*

**Proof.** This is immediate by Gram-Schmidt orthogonalization.  $\square$

**Lemma A.2.** *Let  $k, m \in \mathbb{N}$  with  $m < k$  and  $\mathcal{K}$  an  $m$ -dimensional subspace of  $\mathbb{C}^k$ . Let  $P_{\mathcal{K}}$  be the orthogonal projection onto  $\mathcal{K}$ . Let  $\eta : [0, 1] \rightarrow \mathbb{C}^k$  be a  $C^\infty$ -map with  $\eta(0), \eta(1) \notin \mathcal{K}^\perp$ . Then for any  $\varepsilon > 0$ , there exists a  $C^\infty$ -map  $\xi : [0, 1] \rightarrow \mathbb{C}^k$  such that  $\sup_{t \in [0, 1]} \|\xi(t) - \eta(t)\| < \varepsilon$ ,  $\xi(0) = \eta(0)$ ,  $\xi(1) = \eta(1)$  and  $P_{\mathcal{K}}\xi(t) \neq 0$  for all  $t \in [0, 1]$ .*

**Proof.** This is immediate from Lemma A.1 of [BO] for a sub-manifold  $\mathcal{N} = \mathcal{K}^\perp$ .  $\square$

**Lemma A.3.** *Let  $k \in \mathbb{N}$ ,  $-\infty < a < b < \infty$  and  $t_0 \in [a, b]$ . Let  $P : [a, b] \rightarrow \mathbb{M}_k$  be a  $C^\infty$ -map such that  $P(t) \in \mathcal{P}(\mathbb{M}_k)$  for each  $t \in [a, b]$  and  $\sup_{t \in [a, b]} \|P(t) - P(t_0)\| < \frac{1}{4}$ . Then there exists a  $C^\infty$ -map  $U : [a, b] \rightarrow \mathbb{M}_k$  with  $U(t_0) = 1$  such that  $U(t) \in \mathcal{U}(\mathbb{M}_k)$  and  $P(t) = U(t)P(t_0)U(t)^*$  for each  $t \in [a, b]$ .*

**Proof.** Set  $X(t) := P(t)P(t_0) + (1 - P(t))(1 - P(t_0))$  for  $t \in [a, b]$ . Then we have  $\|X(t) - 1\| < \frac{1}{4}$  and can define  $U(t) := X(t)|X(t)|^{-1}$  as  $X(t)$  is invertible. By definition,  $U(t)$  is unitary for all  $t \in [a, b]$  and satisfies  $U(t_0) = 1$ . Furthermore,  $U : [a, b] \rightarrow \mathbb{M}_k$  is  $C^\infty$ . We claim  $U(t)P(t_0)U(t)^* = P(t)$  for all  $t \in [a, b]$ . To see this, note that  $X(t)P(t_0) = P(t)X(t)$ . From this and its adjoint  $P(t_0)X(t)^* = X(t)^*P(t)$ , we also have  $|X(t)|^{-1}P(t_0) = P(t_0)|X(t)|^{-1}$ . Hence we have  $U(t)P(t_0)U(t)^* = X(t)|X(t)|^{-1}P(t_0)|X(t)|^{-1}X(t)^* = P(t)$ .  $\square$

**Lemma A.4.** *For  $k \in \mathbb{N}$ , let  $P : [0, 1] \rightarrow \mathbb{M}_k$  be a  $C^\infty$ -map such that  $P(t) \in \mathcal{P}(\mathbb{M}_k)$  for each  $t \in [0, 1]$ . Then there exists a continuous and piecewise  $C^\infty$ -map  $U : [0, 1] \rightarrow \mathbb{M}_k$  with  $U(0) = 1$  such that  $U(t) \in \mathcal{U}(\mathbb{M}_k)$  and  $P(t) = U(t)P(0)U(t)^*$  for each  $t \in [0, 1]$ .*

**Proof.** By the continuity of  $P$  and the compactness of  $[0, 1]$ , there exist  $l \in \mathbb{N}$  and  $0 = t_0 < t_1 < \dots < t_l = 1$  such that  $\sup_{t \in [t_i, t_{i+1}]} \|P(t) - P(t_i)\| < \frac{1}{4}$ ,  $i = 0, \dots, l-1$ . By Lemma A.3, there exist  $C^\infty$ -maps  $U_i : [t_i, t_{i+1}] \rightarrow \mathbb{M}_k$ ,  $i = 0, \dots, l-1$  with  $U_i(t_i) = 1$ , such that  $U_i(t) \in \mathcal{U}(\mathbb{M}_k)$ ,  $P(t) = U_i(t)P(t_i)U_i(t)^*$  for  $t \in [t_i, t_{i+1}]$ . Set  $U(t) := U_{l-1}(t)U_{l-2}(t_{l-1}) \dots U_1(t_2)U_0(t_1)$  for  $t \in [t_{l-1}, t_l]$ . Then  $U(t)$  satisfies the required conditions.  $\square$

**Lemma A.5.** *Let  $k, m \in \mathbb{N}$  with  $m < k$ . Let  $\zeta_i : [0, 1] \rightarrow \mathbb{C}^k$ ,  $i = 1, \dots, m$  be  $C^\infty$ -maps. For each  $t \in [0, 1]$  let  $P(t)$  be the orthogonal projection onto  $\text{span}\{\zeta_i(t)\}_{i=1}^m$ . Suppose that for each  $t \in [0, 1]$ , the vectors  $\{\zeta_i(t)\}_{i=1}^m$  are linearly independent. Let  $\xi_0, \xi_1 \in \mathbb{C}^k$  with  $(1 - P(0))\xi_0, (1 - P(1))\xi_1 \neq 0$ . Then there exists a continuous and piecewise  $C^\infty$ -map  $\xi : [0, 1] \rightarrow \mathbb{C}^k$  with  $\xi(0) = \xi_0$ ,  $\xi(1) = \xi_1$  such that  $(1 - P(t))\xi(t) \neq 0$  for all  $t \in [0, 1]$ .*

**Proof.** By Lemma A.1,  $[0, 1] \ni t \mapsto P(t) \in \mathbb{M}_k$  is  $C^\infty$ . Therefore, applying Lemma A.4 to  $P(t)$  we obtain a continuous and piecewise  $C^\infty$ -map  $U : [0, 1] \rightarrow \mathcal{U}(\mathbb{M}_k)$  with  $U(0) = 1$  such that  $P(t) = U(t)P(0)U(t)^*$  for each  $t \in [0, 1]$ . Set  $\eta_0 = \xi_0$ , and  $\eta_1 := U(1)^*\xi_1$ . By the assumptions,  $(1 - P(0))\eta_0 \neq 0$  and  $(1 - P(0))\eta_1 \neq 0$ . Let  $\tilde{\eta}(t) := (1 - t)\eta_0 + t\eta_1$ . Applying Lemma A.2 to

this  $\tilde{\eta}$  and  $(1 - P(0))\mathbb{C}^k$  we obtain a  $C^\infty$ -map  $\tilde{\xi} : [0, 1] \rightarrow \mathbb{C}^k$  such that  $\tilde{\xi}(0) = \eta_0$ ,  $\tilde{\xi}(1) = \eta_1$  and  $\mathcal{K} := (1 - P(0))\tilde{\xi}(t) \neq 0$  for all  $t \in [0, 1]$ . Set  $\xi(t) := U(t)\tilde{\xi}(t)$  for  $t \in [0, 1]$ . Then  $\xi : [0, 1] \ni t \mapsto \xi(t) \in \mathbb{C}^k$  is continuous and piecewise  $C^\infty$ . Furthermore, we have

$$(1 - P(t))\xi(t) = U(t)(1 - P(0))U(t)^*U(t)\tilde{\xi}(t) = U(t)(1 - P(0))\tilde{\xi}(t) \neq 0,$$

and  $\xi(0) = \xi_0$ ,  $\xi(1) = \xi_1$ . □

The proof of the following Lemma is standard.

**Lemma A.6.** *Let  $k, m \in \mathbb{N}$  with  $m < k$ . Let  $X : [0, 1] \rightarrow (\mathbf{M}_k)_+$  be continuous and piecewise  $C^\infty$ -path of positive matrices such that the rank of  $X(t)$  is  $m$  for all  $t \in [0, 1]$ . Let  $S(t)$  be the support projection of  $X(t)$ , and set  $\gamma(t) := d_{\mathbb{R}}(\sigma(X(t)) \setminus \{0\}, \{0\})$ . Then, the path of projections*

$$S : [0, 1] \ni t \mapsto S(t) \in \mathbf{M}_k$$

*is continuous and piecewise  $C^1$  and*

$$\inf_{t \in [0, 1]} \gamma(t) > 0.$$

## B CP maps of matrix algebras

In this section we collect known results about positive maps on matrix algebras.

**Lemma B.1.** *Let  $n, k \in \mathbb{N}$  and  $\mathbf{v} \in \mathbf{M}_k^{\times n}$ . Suppose that there exist  $l_0, d \in \mathbb{N}$ , such that*

- (i)  $\mathcal{K}_{l_0}(\mathbf{v})$  has an invertible element, and
- (ii) there exists  $l_1 \in \mathbb{N}$  such that  $\dim \mathcal{K}_l(\mathbf{v}) = d$  for all  $l \geq l_1$ .

*Then we have*

$$\min \{l \in \mathbb{N} \mid \dim \mathcal{K}_{l'}(\mathbf{v}) = d, \text{ for all } l' \geq l\} \leq dl_0.$$

**Proof.** The proof is basically in [SPWC]. □

**Lemma B.2.** [SPWC] *Let  $n, n_0 \in \mathbb{N}$  and  $\omega \in \text{Prim}(n, n_0)$ . Define*

$$l_\omega := \inf \{l \in \mathbb{N} \mid \mathcal{K}_{l'}(\omega) = \mathbf{M}_{n_0}, \text{ for all } l' \geq l\}.$$

*Then we have  $l_\omega \leq n_0^4$ .*

The following statement is standard.

**Lemma B.3.** *Let  $k \in \mathbb{N}$  and  $T : [0, 1] \rightarrow B(\mathbf{M}_k)$  be a  $C^\infty$ -map satisfying the following conditions for each  $t \in [0, 1]$ :*

- (1) *The spectral radius  $r_{T(t)}$  is strictly positive, and*
- (2) *the spectral radius  $r_{T(t)}$  is a non degenerate eigenvalue of  $T(t)$ , and*
- (3) *there exists  $0 < s_t < r_{T(t)}$  such that  $\sigma(T(t)) \setminus \{r_{T(t)}\} \subset \mathcal{B}_{s_t}(0)$ .*

*Then*

- 1. *the map  $[0, 1] \ni t \mapsto P_{\{r_{T(t)}\}}^{T(t)} \in B(\mathbf{M}_k)$  is  $C^\infty$ , and*
- 2.  *$[0, 1] \ni t \mapsto r_{T(t)} \in \mathbb{C}$  is  $C^\infty$ , and*
- 3. *there exists  $0 < s < 1$  such that  $\sigma(r_{T(t)}^{-1}T(t)) \setminus \{1\} \subset \mathcal{B}_s(0)$ , for all  $t \in [0, 1]$ .*



## C Linear spaces spanned by given set of vectors

**Lemma C.1.** Let  $m, k \in \mathbb{N}$ , with  $m \leq k$ , and  $\{\xi_i\}_{i=1}^m$ , a set of vectors of  $\mathbb{C}^k$ . Let  $A$  be an  $m \times m$  matrix given by  $A = (\langle \xi_i, \xi_j \rangle)_{i,j=1}^m$ . Let  $X := \sum_{i=1}^m |\xi_i\rangle \langle \xi_i| \in M_k$  and  $P$  be the support projection of  $X$ . Suppose that there exists a positive constant  $c$  such that  $c\mathbb{I} \leq A$ . Then we have  $cP \leq X$ . In particular, we have  $\sigma(X) \setminus \{0\} \subset [c, \|X\|]$ .

**Lemma C.2.** Let  $0 < a_1 < a_2$ . There is a constant  $C_{a_1, a_2}, C'_{a_1, a_2} > 0$  which depend only on  $a_1, a_2$  satisfying the following.: Let  $k, m \in \mathbb{N}$  with  $m \leq k$ ,  $\{\xi_i\}_{i=1}^m, \{\eta_i\}_{i=1}^m$  sets of linearly independent vectors in  $\mathbb{C}^k$ . Let  $P, Q$  be orthogonal projections onto the subspace spanned by  $\{\xi_i\}, \{\eta_i\}$ , respectively. Assume that the spectrum of  $X = \sum_{i=1}^m |\xi_i\rangle \langle \xi_i|, Y = \sum_{i=1}^m |\eta_i\rangle \langle \eta_i| \in M_k$  satisfy  $\sigma(X) \setminus \{0\}, \sigma(Y) \setminus \{0\} \subset (a_1, a_2)$ , and  $\|X - Y\| < C_{a_1, a_2}$ . Then we have

$$\|P - Q\| \leq C'_{a_1, a_2} \|X - Y\|.$$

## D Quasi-equivalence of states

**Lemma D.1.** Let  $d \in \mathbb{N}$ , and  $\mathfrak{A}$  be a unital  $C^*$ -algebra. Suppose that for any  $N \in \mathbb{N}$ , there exists a unital  $C^*$ -algebra  $\mathfrak{B}_N$  such that  $\mathfrak{A} = \left(\bigotimes_{i=1}^N M_d\right) \otimes \mathfrak{B}_N$ . Let  $\varphi_1, \varphi_2$  be states on  $\mathfrak{A}$ . Suppose that there exist constants  $C > 0, 0 < s < 1$  such that

$$|\varphi_1(A) - \varphi_2(A)| \leq Cs^N \|A\|, \quad N \in \mathbb{N}, \quad A \in \mathbb{I}_{\bigotimes_{i=1}^N M_d} \otimes \mathfrak{B}_N.$$

Then  $\varphi_1$  and  $\varphi_2$  are quasi-equivalent.

**Proof.** The proof is analogous to that of Lemma 6.2.55 of [BR96]. We omit the details.  $\square$

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